Autumn Quarter 2005
Math. Methods Problem Set 2: Iterated maps

October 27, 2005

1 Analysis of a new iterated map

Consider the system defined by $x_{n+1} = f(x_n, a)$, where

$$f(x, a) = ax \cos(x) \tag{1}$$

You are to perform a complete analysis of the behavior of this system. Using a `mapExplorer` object, and the `orbitDiagram.py` script will help you in your exploration. This exploration requires a combination of analytic work (i.e. using mathematics on paper) and computer simulations. Note that in some cases, you may not be able to find an analytic expression for some of the answers you need (e.g. the maxima of the function $x \cos(x)$). In such cases, feel free to get an approximation by writing a Python script that helps you get an answer, e.g. by printing out a table of values or using a Newton Method iteration.

The following provides some guidelines for your exploration. In all cases, it is implicit that you should discuss how things change as the parameter a is varied.

- Where are the fixed points? Discuss their stability

- If x is initially in the interval $[0, \frac{\pi}{2}]$ under what circumstances will it stay there? What can you say if x is outside this interval?

- When does a stable period-2 orbit appear? When does this go unstable? Discuss what kinds of other periodic orbits appear after the period-2 orbit becomes unstable

- When does the system appear to have a dense set of unstable periodic orbits?

- Show that this system exhibits "unpredictability," once it goes chaotic, in the sense that initially close orbits diverge. Estimate the Lyapunov exponent. Show an example of a chaotic orbit.

Solution: (Run the accompanying Python script to see the accompanying figures).

Note that I have generalized some of the results to the entire positive Real axis, though I am only expecting that the student will have explored the dynamics in the interval $[0, \pi/2]$.

Fixed points: Set $f(x) = x$. There is always a fixed point at x=0. $f'(0) = a$, so this fixed point is stable if and only if $|a| < 1$. The other fixed points satisfy $a \cos x = 1$, i.e. $x = \cos^{-1}(a^{-1})$. There are an infinite number of these on the real axis if $|a| \geq 1$. Thus, the additional fixed points appear precisely when the fixed point at $x = 0$ goes unstable. By looking at a plot of $y = f(x)$ on the same graph as a plot of $y = x$ you can see that this isn't accidental. If $f'(0) > 1$, then since the slope starts out greater than the slope of $y = x$, and since $f(\pi/2) = 0$, it is inevitable that the curve will cross the line $y = x$ at some positive value.

To determine the stability of the rest of the fixed points we evaluate $f'(X_j)$, where X_j is the jth fixed point. Let X_0 be the nonzero fixed point which lies in the interval $[0, \pi/2]$. For a just slightly above 1, X_0 is close to zero. As a gets large X_0 approaches $\pi/2$. The rest of the fixed points are at $X_0 + 2\pi, -X_0 + 2\pi, X_0 + 4\pi, -X_0 + 4\pi, ...$ Now,

$$f'(X_j) = a \cos X_j - a X_j \sin X_j = 1 - X_j \sqrt{a^2 - 1} \qquad (2)$$

The positive fixed points are then stable when $X_j \sqrt{a^2 - 1} < 2$. For any j, this is always satisfied for a sufficiently close to 1, but the larger j is, the closer a has to be to 1 for the fixed point to be stable. For the first positive fixed point, the fixed point is unstable for $a > 2.1093$. For the second, the stability boundary is $a = 1.0129$ and for the third it is at $a = 1.0123$. You can get these stability boundaries by using Newton's method to solve $(\pm \cos^{-1}(a^{-1}) + 2\pi j)\sqrt{a^2 - 1} = 2$

Confinement of the orbit: To find out when the orbit stays in the interval $[0, \frac{\pi}{2}]$, we maximize $f(x)$ over this interval. The maximum occurs

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where $0 = f'(x) = a \sin x - ax \cos x$, i.e. where $\tan x = x^{-1}$ independent of a. Denoting this point by x_m, the maximum value is

$$f(x_m) = ax_m \cos(x_m) = \pm \frac{ax_m^2}{\sqrt{x_m^2 + 1}} \qquad (3)$$

The sign on the right hand side alternates from one interval to the next, according to whether the point is a maximum or a minimum. There is one maximum or minimum on each of the subintervals between the zeros of $\cos x$. We find the first positive maximum by applying Newton's method to $x \tan x - 1$; it is at .8603, where the value of f is .561096. Hence the orbit stays in the subinterval as long as $a < (\pi/2)/.561096 = 2.799$.

For the subsequent maxima/minima x_m becomes large, $1/x_m$ becomes small, and so the extrema are located quite near to the zeros of $\tan(x)$, namely $x = j\pi$. The values of f near at these maxima are approximately $\pm aj\pi$. This is already a quite good approximation for the first positive minimum and the second positive maximum; one can do better by doing a Taylor series expansion of tan about $j\pi$. This is left to the student.

If $x > \pi/2$ initially, then a must be smaller to guarantee confinement. If x initially exceeds $\pi/2$, the next iterate can become negative, but can the orbit escape to infinity? For example, if $-3\pi/2 < x < 3\pi/2$, the maximum possible value at the next iterate has $|x| < a\pi$, using the lowest order approximation to the maxima/minima in this interval. Hence, the orbit remains in the interval if $a\pi < 3\pi/2$, i.e. $a < 3/2$. In this range, the map has a stable fixed point only in the smaller sub-interval $[-\pi/2, \pi/2]$. It has unstable period-2 orbits in the broader interval, but by playing around with `plotOrbit`, it appears that almost all other initial conditions in the large interval eventually are attracted to a stable fixed point in the smaller interval. On the other hand, when $a > 3/2$, the solutions can run off to infinity. It is left to the student to generalize the result to larger intervals.

Here's something else interesting to explore: When a is slightly above 2.799 it is possible to have orbits starting in the interval $[-\pi/2, \pi/2]$ which make transitions between positive and negative values. For what range of a are such orbits "typical" for initial conditions in the interval? Do orbits eventually escape the interval? For cases where the orbits do not blow up, explore the statistics of transitions between positive and negative values. What is the probability that the orbits remain positive for N consecutive iterations? How does this persistence probability vary with a?

Periodic Orbits: We suspect a period-2 orbit is born just above the value of a where the fixed point goes unstable. We verify this using the `plotComposition` method with $a = 2.2$. By plotting an orbit, we see that the period-2 orbit we have found is stable. (This is also evident from the slope in the composition graph). See the Python script for details and output.

Determining the stability boundary for the period-2 orbit analytically is intractible, so we use the `mapExplorer` and `plotOrbit` to hunt around and see when the orbit goes unstable. This happens near $a = 2.39$. The stability can also be determined using the `findPeriodicOrbits` method. By using `plotComposition`, we see that a stable period-4 orbit is introduced at this point.

Onset of dense periodic orbits: We run the `orbitDiagram` function to look at the introduction of new periodic orbits as a is increased. We vary a from the value where the period-2 orbit goes unstable, to the maximum value that keeps the orbit confined to the interval. From looking at the result, we see that the set of periodic orbits start to look dense somewhere between $a = 2.5$ and $a = 2.6$. The orbits are not dense in the whole interval; there are gaps.

Chaotic orbits: We set $a = 2.7$, which is well within the range where periodic orbits appear to be dense. As a first demonstration, we plot two orbits on the same graph, starting from slightly different initial conditions. Then, we plot a graph of the log of the difference. The difference grows until it is comparable to the width of the interval, and then it can grow no more. By estimating the slope during the exponentially growing stage, we find that the Lyapunov exponent – the exponential rate of separation of trajectories – is about .47. That means that an initially small error grows like $\exp .47n$.

The moral of the story: In the range of a where the orbits are confined to the interval $[0, \pi/2]$ the overall picture we see for this map is rather similar to what we saw for the logistic map. There is a sequence of period-doubling bifurcations, leading to a dense set of periodic orbits, and then chaos. The chaotic orbits are characterized by exponential separation of initially close orbits. This all suggests that all "one-hump" maps produce behavior which is, in some sense, the same. It gives us some common things to look for in chaotic systems.

2 Programming project: Computing a histogram

Solution: See accompanying Python script.