Autumn Quarter 2005
Math. Methods Problem Set 5

November 11, 2005

1 1D Linear Inhomogeneous and Homogeneous equations

Consider a population which grows during the day but dies out at night. There is also a migration of new individuals into the region at a constant rate. The equation governing this system is

$$\frac{dP}{dt} = (g_o \sin t)P + a \tag{1}$$

where g_o and a are constants.

(a) Find the homogeneous solution of this equation. Discuss its behavior.

(b) Write down an expression for a particular solution of this equation, satisfying $P(0) = 0$, in terms of a definite integral over t. You probably will not be able to carry out the integral analytically.

(c) Use the Trapezoidal Rule with Romberg extrapolation to evaluate the integral of Part (b) at $t = 0, 2\pi, 4\pi, ..., 20\pi$ numerically. Is the long term population trend growth or decay? How rapid is the growth or decay? (e.g. linear vs. exponential). Try to infer this behavior directly from the integral in Part (b)

2 Second order linear, homogeneous equations with const. coefficients

Consider the equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0 \tag{2}$$

(a) Write down the characteristic polynomial, and find the two fundamental solutions.

(b) Find the superposition of the two solutions that satisfies the initial condition $x = 1, \frac{dx}{dt} = 1$ at $t = 0$.

(c) Write a Python script to plot this orbit in the $(x, \frac{dx}{dt})$ plane.

(d) Do the same for the equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0 \tag{3}$$

subject to the same initial conditions.

3 Linearization in 2D

Consider a population of rabbits (denoted by r) and wolves (denoted by w). You can think of r as the mass of rabbits per square kilometer, and w as the mass of wolves per square kilometer. The rabbits live off of grass, and in the absence of wolves their population would be limited to a carrying capacity C, limited by the grass supply. The wolves live only off of rabbits (poor bunnies!). In the absence of rabbits, the wolf population would die off at a rate d. With rabbits as food (poor bunnies! lucky wolves!) the wolf population grows exponentially with a growth rate proportional to the bunny population. This system is described by the equations

$$\frac{dr}{dt} = g_r \cdot (1 - \frac{r}{C})r - e \cdot r \cdot w; \frac{dw}{dt} = g_w \cdot r \cdot w - d \cdot w \tag{4}$$

In this equation, e is the eating rate, which will not generally be the same as the growth rate of wolf biomass, since rabbit mass is not converted completely into wolf mass; there is some wastage. To be physically consistent, though, the "conversion efficiency" g_w/e ought to be less than unity. These equations are a slight generalization of the *Lotka-Volterra Predator-Prey Equations.* The generalization consists in adding a carrying capacity for the prey population.

Find all the equilibrium points of this system. Linearize the system about each of the equilibrium points, and write down the matrices determining the stability of the equilibrium points. Which are stable and which are unstable?