Autumn Quarter 2005
Math. Methods Problem Set 5

November 29, 2005

1 1D Linear Inhomogeneous and Homogeneous equations

Consider a population which grows during the day but dies out at night. There is also a migration of new individuals into the region at a constant rate. The equation governing this system is

$$\frac{dP}{dt} = (g_o \sin t)P + a \tag{1}$$

where g_o and a are constants.

(a) Find the homogeneous solution of this equation. Discuss its behavior.

Solution:

$$\frac{1}{P}\frac{dP}{dt} = (g_o \sin t) \tag{2}$$

so

$$\ln P = g_o \cos t + const.; P = Ae^{-g_o \cos t} \tag{3}$$

The solution is periodic. It grows to a maximum at $t = \pi$, decays back to its original value at $t = 2\pi$, and repeats the pattern.

(b) Write down an expression for a particular solution of this equation, satisfying $P(0) = 0$, in terms of a definite integral over t. You probably will not be able to carry out the integral analytically.

Solution: Write $P = A(t)P_h(t)$ where P_h is a solution of the homogeneous equation. Then, substituting in the original equation

$$\frac{dA}{dt} = a/P_h(t) = ae^{g_o \cos t} \tag{4}$$

so

$$A(t) = a \int_0^t e^{g_o \cos t'} dt' \tag{5}$$

and

$$P(t) = a \int_0^t e^{g_o(\cos t' - \cos t} dt' \tag{6}$$

Note that the integrand is positive, and periodic in t' with period 2π. Therefore, the $P(t+2\pi) = P(t) + P(2\pi)$, and hence the population grows linearly with time, with a oscillations superposed.

(c) Use the Trapezoidal Rule with Romberg extrapolation to evaluate the integral of Part (b) at $t = 0, 2\pi, 4\pi, ..., 20\pi$ numerically. Is the long term population trend growth or decay? How rapid is the growth or decay? (e.g. linear vs. exponential). Try to infer this behavior directly from the integral in Part (b)

Solution From the discussion of the previous part, we only need to evaluate $P(t)$ in the interval $[0, 2\pi]$. This is done in the accompanying Python script. Note that we are free to take units such that $a = 1$ and $g_o = 1$, since the former just multiplies the solution by a constant and the latter can be absorbed in the definition of time.

2 Second order linear, homogeneous equations with const. coefficients

Consider the equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0 \tag{7}$$

(a) Write down the characteristic polynomial, and find the two fundamental solutions.

Solution: $\lambda^2 + \lambda + 1 = 0$, i.e. $\lambda = \frac{1}{2}(-1 \pm \sqrt{3}i)$ The fundamental solutions can be written as exponentials:

$$e^{-t/2} e^{\pm i\sqrt{3}t/2} \tag{8}$$

or, by adding and subtracting the two complex exponentials, as

$$e^{-t/2} \sin(\sqrt{3}t/2), e^{-t/2} \cos(\sqrt{3}t/2) \tag{9}$$

(b) Find the superposition of the two solutions that satisfies the initial condition $x = 1, \frac{dx}{dt} = 1$ at $t = 0$.

Solution: Let's work with the real form of the two fundamental solutions. The solution proportional to sin vanishes at $t = 0$ so we write immediately

$$x(t) = e^{-t/2}(\cos(\sqrt{3}t/2) + A\sin(\sqrt{3}t/2)) \tag{10}$$

Then taking the time derivative and setting it equal to unity at $t = 0$ we find $-\frac{1}{2} + \sqrt{3}/2A = 1$, i.e. $A = \sqrt{3}$. The velocity is

$$e^{-t/2}((-\frac{1}{2} + A\sqrt{3}/2)\cos(\sqrt{3}t/2) + (-\frac{1}{2} - \sqrt{3}/2)\sin(\sqrt{3}t/2) \tag{11}$$

As an exercise, you can try doing this using the complex exponentials instead, to see which way you find easier.

(c) Write a Python script to plot this orbit in the $(x, \frac{dx}{dt})$ plane. *Solution:* See accompanying Python script.

(d) Do the same for the equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0 \tag{12}$$

subject to the same initial conditions.

Solution: The characteristic polynomial is $\lambda^2 + 2\lambda + 1 = 0$, which has a double root at $\lambda = -1$. The two fundamental solutions are then

$$e^t, te^{-t} \tag{13}$$

To satisfy the initial condition on x we write $x(t) = (1+At)e^{-t}$. Then, taking the derivative we find $A = 2$. The velocity is $v = (A - 1 - At)e^{-t}$. We plot in the accompanying Python script as before.

3 Linearization in 2D

Consider a population of rabbits (denoted by r) and wolves (denoted by w). You can think of r as the mass of rabbits per square kilometer, and w as the mass of wolves per square kilometer. The rabbits live off of grass, and in the absence of wolves their population would be limited to a carrying capacity C, limited by the grass supply. The wolves live only off of rabbits (poor bunnies!). In the absence of rabbits, the wolf population would die off at a rate d. With rabbits as food (poor bunnies! lucky wolves!) the wolf population grows exponentially with a growth rate proportional to the bunny population. This system is described by the equations

$$\frac{dr}{dt} = g_r \cdot (1 - \frac{r}{C})r - e \cdot r \cdot w; \frac{dw}{dt} = g_w \cdot r \cdot w - d \cdot w \tag{14}$$

In this equation, e is the eating rate, which will not generally be the same as the growth rate of wolf biomass, since rabbit mass is not converted completely into wolf mass; there is some wastage. To be physically consistent, though, the "conversion efficiency" g_w/e ought to be less than unity. These equations are a slight generalization of the *Lotka-Volterra Predator-Prey Equations.* The generalization consists in adding a carrying capacity for the prey population.

Find all the equilibrium points of this system. Linearize the system about each of the equilibrium points, and write down the matrices determining the stability of the equilibrium points. Which are stable and which are unstable?

Solution: The linearized system is

$$\frac{d}{dt}\begin{bmatrix}\delta r\\ \delta w\end{bmatrix} = \begin{bmatrix} g_r \cdot (1 - 2\frac{r_o}{C}) - e w_o & -e r_o \\ g_w w_o & g_w r_o - d \end{bmatrix}\begin{bmatrix}\delta r\\ \delta w\end{bmatrix} \tag{15}$$

where (r_o, w_o) is an equilibrium point. The three equilibria are: $r_o = w_o = 0$ (no rabbits or wolves $r_o = C, w_o = 0$ (rabbits at maximum capacity, but no wolves to eat them (lucky bunnies!), and $r_o = d/g_w$, $w_o = (g_r/e) \cdot (1 - \frac{r_o}{C})$ (rabbits and wolves in not-so-peaceful coexistence).

By plugging into the matrix and finding the eigenvalues, we find that the first of these three solutions has the matrix

$$\begin{bmatrix} g_r & 0 \\ 0 & 0 \end{bmatrix} \tag{16}$$

and so is unstable (a small introduced rabbit population grows exponentially at rate g_r). The second solution has the matrix

$$\begin{bmatrix} -g_r & -eC \\ 0 & g_w C - d \end{bmatrix} \tag{17}$$

which has one stable eigenvalue ($\lambda = -g_r$) and a one eigenvalue which is unstable if $g_w C - d > 0$. If this eigenvalue is unstable, there is sufficient rabbit population to support a growing wolf population. The third equilibrium has the matrix

$$\begin{bmatrix} -g_r\frac{r_o}{C} & -er_o \\ g_w w_o & 0 \end{bmatrix} \tag{18}$$

The eigenvalues are given by

$$\lambda(\lambda + g_r\frac{r_o}{C}) + eg_w r_o w_o = 0 \tag{19}$$

from which the stability can be easily determined. Note that we can reduce the parameter space because we are free to adopt units of time such that $g_r = 1$ and units of mass such that $C = 1$. Then, the free parameters are g_w, e and d. The characteristic polynomial can then be written

$$\lambda(\lambda + r_o) + g_w r_o(1 - r_o) = 0 \tag{20}$$

The solutions are

$$2\lambda = -r_o \pm \sqrt{(1 + 4g_w)r_o^2 - g_w r_o} \tag{21}$$

Note that the eating rate e doesn't affect the stability. When r_o is large, the + branch yields a growing solution, so the equilibrium is unstable. When r_o is small, the equilibrium is stable,but the eigenvalues form a complex conjugate pair, and a perturbation spirals back toward the equilibrium point.