Paris Math Problem Comments

February 3, 2005

1 Radiative cooling

The problem posed was the solution of

$$\frac{dT}{dt} = 1 - T^4 \tag{1}$$

This is done by partial fractions. Write

$$-\frac{1}{T^4 - 1} dT = dt \tag{2}$$

then note that the four roots of the denominator on the lhs are ± 1 and $\pm i$. Therefore, the fraction can be written in the form

$$\frac{1}{T^4 - 1} = \frac{a}{T - 1} + \frac{b}{T + 1} + \frac{c}{T - i} + \frac{d}{T + i} \tag{3}$$

To find the coefficients, multiply both sides by $T^4 - 1$, yielding

$$1 = a(T+1)(T-i)(T+i) + b(T-1)(T-i)(T+i) + c(T-1)(T+1)(T+i) + d(T-1)(T+1)(T-i) \tag{4}$$

The method for finding the coefficients in a problem like this, without getting involved in a ridiculous amount of algebra, is to note that the equation must hold for whatever values of T we plug in. If we plug in a root of $T^4 - 1$, all terms on the right hand side vanish except one, giving us a simple expression for one coefficient at a time. For example, plugging in $T = 1$ yields the equation $1 = a(1 + 1)(1 - i)(1 + i)$ from which we infer immediately that $a = 1/4$. Doing this for the other roots gives us $b = -1/4$, $c = i/4$, $d = -i/4$.

Substituting Eqn. 3 into Eqn. 2, integrating, and substituting in the values of the coefficients, yields:

$$\ln[(\frac{T+1}{T-1})^{1/4}(\frac{T+i}{T-i})^{i/4}] - \ln[(\frac{T_o+1}{T_o-1})^{1/4}(\frac{T_o+i}{T_o-i})^{i/4}] = t \tag{5}$$

The second term on the lhs allows us to satisfy the initial condition $T(0) = T_o$. This completes the solution in the sense that we now know the function $t(T)$. Let's go a bit further, though, and deduce some properties of the solution. Let

$$F(T) = \ln(\frac{T+1}{T-1}\frac{T_o-1}{T_o+1})^{1/4} \tag{6}$$

and

$$G(T) = \ln(\frac{T+i}{T-i}\frac{T_o-i}{T_o+i})^{i/4} \tag{7}$$

so that the solution becomes $F(T)+G(T) = t$. The behavior of $F(T)$ is easy to discern: $F(T_o) = 0$, and F increases mononotonically to infinity as $T \rightarrow 1$ from either above or below. Understanding G takes just a little more work. Write $T + i = Aexp(i\theta)$ and analogously for T_o. Then, we find

$$G(T) = \ln e^{(2i(\theta-\theta_o))(i/4)} = -\frac{1}{2}(\theta - \theta_o) \tag{8}$$

As required, G is real. $G(T)$ starts at zero at $T = T_0$, and monotonically approaches the value $-\frac{1}{2}(\frac{\pi}{4} - \theta_o)$ as $T \rightarrow 1$ from either above or below.

The net result is that $F(T) + G(T)$ becomes positively infinite as $T \rightarrow 1$, regardless of T_o (assuming $T_o > 0$), which has the consequence that $T(t) \rightarrow 1$ as $t \rightarrow \infty$. The solution is plotted for initial conditions with $T_o < 1$ and with $T_o > 1$ in Figure 1, using the trick (or technique, if you will) of plotting $t(T)$ and interchanging the axes, so that we never have to solve explicitly for $T(t)$.

There are two limits in which an approximate solution to Eqn. 1 can be obtained with comparatively little work, and which can be used to check the limiting behavior in the full solution. The first limit is $T_o >> 1$, in which case the T^4 term on the rhs of Eqn. 1 overwhelms the constant term, until such time as T decreases enough to be order unity. Until that happens, the solution is approximately $T = 1/(3t + 1/T_o^3)^{\frac{1}{3}}$. It can be shown that the exact solution reduces to this for large T, but it takes a fair amount of work to do so, since $F(T) + G(T)$ needs to be expanded out to third order in $1/T$. This is left as an exercise to the energetic reader.

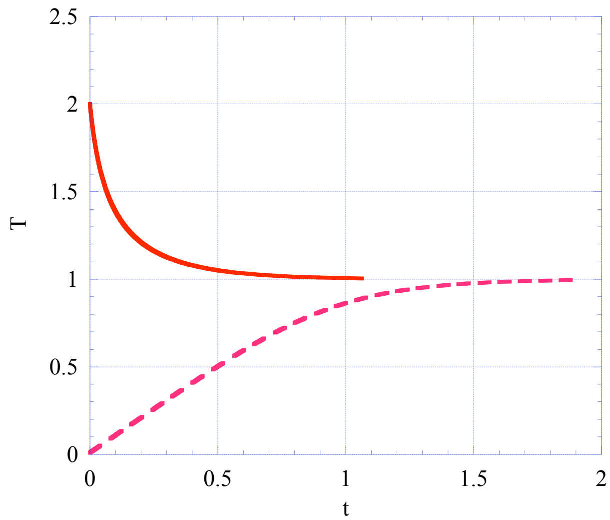

Figure 1: Exact solution to radiative cooling problem, for two different initial conditions.

The other limit which yields an easy solution is that in which the solution starts near the fixed point $T = 1$. Writing $T = 1 + T'$, with T' presumed small, the differential equation becomes $dT'/dt = -4T'$, whence $T' = T'_o exp(-4t)$. In other words, the fixed point is stable, and any deviation from the fixed point relaxes exponentially to zero.

For $T \approx 1$, $F + G \approx -\frac{1}{2}(\frac{\pi}{4} - \theta_o) - \frac{1}{4}\ln((T-1)/(T_o-1))((T_o+1)/2)$ so the exact solution reduces to

$$T - 1 = (T_o - 1)\frac{2}{T_o + 1}e^{-(\frac{\pi}{2} - 2\theta_o)}e^{-4t} \tag{9}$$

near the fixed point. The behavior of this agrees with the solution obtained directly by linearizing about the fixed point, though the latter method does not give us the coefficient of the decaying exponential which corresponds to a given (arbitrarily large) initial condition. The coefficient does reduce to T'_o when the initial condition is near the fixed point, though.

Figure 2 compares the two approximate solutions with the exact solution proceeding starting from $T = 2$. We see that the two limiting solutions give a good picture of the full behavior of the system. This is an object-lesson for systems which can't be solved exactly: look for many limiting cases which can be solved, and try to piece together the the behavior from the approximate solutions.

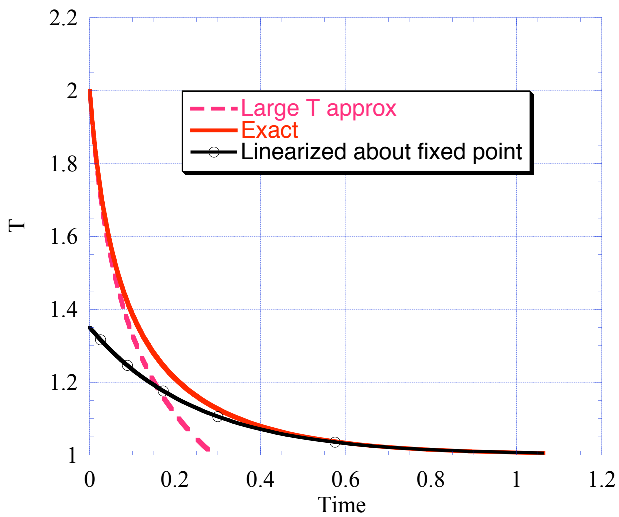

Figure 2: Approximate and exact solutions of the approach to radiative equilibrium