Paris Math Problem Set 2

January 12, 2005

1 Convergence of Taylor series for analytic functions

Develop the Taylor series for the function $f(z) = 1/(1 + z^2)$. (Hint: Expand as a series in z^2). When does the series converge? Write a Python script to check your result for values both on and off the real axis. Interpret your result in terms of the position of the poles of the function.

2 The zeta function

The Riemann zeta function is defined by the sum

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{1}$$

where s is a complex number with $\Re(s) > 1$. The definition can be extended to the rest of the plane by *analytic continuation*, but that will not concern us here. Write a Python function to compute the n-term approximation $\zeta_n(s)$. For your first attempt, implement the function as a straightforward loop. Then, implement a second version of the function using `Numeric` array arithmetic and perhaps the `Numeric.fromfunction(...)` method, together with `Numeric.sum(...)` which computes the sum of the elements of the array given as its argument. Compare how long the two different versions take to run, for large m.

As a function of s, approximately how large an n do you have to take to achieve a given accuracy.

Make some plots of an approximation to $\zeta(s)$ along the lines $s = a + yi$ with a held constant (greater than unity). Does ζ appear to have any zeroes in this region?

Extra credit: Show that the only zeroes of ζ for non-negative $\Im(s)$ lie on the line $\Im(s) = 1/2$.

3 Radiative relaxation toward equilibrium

The differential equation for temperature evolution in the presence of radiative cooling and constant solar heating is

$$M\frac{dT}{dt} = S - \sigma T^4 \tag{2}$$

where M is the heat capacity of the surface (a positive constant), S is the solar heating rate (a positive constant) and σ is the Stefan-Boltzmann constant.

What are the equilibrium point(s) of this equation? Will a solution displaced from equilibrium tend to return toward equilibrium? What is the approximate behavior of the solution when the initial temperature is much greater than the equilibrium temperature?

Find the full solution to the equation using partial fractions. It is not possible to find an analytic expression for $T(t)$, but you can find an expression for $t(T)$, which you can plot. Plot the results and discuss the behavior. *Hint:* You will need to go into the complex plane to factor the denominator, even though the sum of the partial fractions will be real. The fourth roots of unity are $u_n = exp(in2\pi/4)$ for $n = 0, 1, 2, 3$.

4

An autonomous ODE is one in which the independent variable does not appear explicitly in the equation (except in the derivative), e.g. $dy/dx = f(y)$. Show that an autonomous first-order ODE cannot oscillate, i.e. that the solution $y(x)$ can only cross any given line $y = const.$ a single time.

5

Find the Green's function for $y'' - y$ for x in $(-\infty, \infty)$, satisfying the condition that the solution vanish at both large and small x. Recall that the Green's function satisfies $d^2 G(x, x_o)/dx^2 - G = \delta(x - x_o)$, and that the equation implies that G is continuous at x_o. Show there is no Green's function for this problem which satisfies $G = 0$ for $x < x_o$, if we also require that G vanishes at large x.

6

Show that every solution of the ODE $y'' + q(x)y = 0$ has infinitely many zeros in $(1, \infty)$ if $q(x) > B/x^2$ with $B > \pi^2$.

Note: You can actually do much better than this. The equation $y'' + (B/x^2)y$ can be solved exactly by tranforming to a Ricatti equation, which allows you to replace the condition by the much sharper condition $B > \frac{1}{4}$, and to make the proposition "if and only if." (Birkhoff and Rota 2d edition problems, p. 57).