Paris Math Problem Set, Week 3

January 19, 2005

1 Green's function for first-order linear ODE

Find the Green's function for the first order linear inhomogeneous problem $dy/dx + q(x)y = r(x)$. That is, find the solution to

$$\frac{d}{dx}G(x, x1) + q(x)G(x, x1) = \delta(x - x_1) \tag{1}$$

Note that in this case, G will not be continous across x_1. Integrate the equation over a small neighborhood of x_1 to obtain the jump condition on G.

Then, use the Green's function to write down a general solution for y(x) in terms of an integral involving r(x).

This general solution is more commonly derived using the method of "variation of parameters" (see Chapter 1 of Birkhoff and Rota).

2 Green's function for the undamped harmonic oscillator

Find the Green's function that solves the initial value problem for the equation $d^2y/dx^2 + y = r(x)$. By "initial value problem," we mean that the solution should have the property that it starts in a state of rest before any force is applied. In other words, if $r(x) = 0$ for $x < x_o$, then $y(x) = 0$ for $x < x_o$. In terms of the Green's function, this translates into the requirement that $G(x, x_1) = 0$ for $x < x_1$ (*Caution:* x_1 is just the second argument of the Green's function here. It has nothing to do with x_o).

Optional: Note that this construction does not prevent us from using the Green's function for solving problems where the "forcing" $r(x)$ extends to $-\infty$. Use your Green's function to find the oscillation you are left with for $x > 0$ if $r(x) = e^{ax}/a$ for all $x < 0$ and $r(x) = 0$ otherwise, with $a > 0$. Discuss and interpret the way the amplitude of the oscillation depends on a.

3 Asymptotic series

Develop the complete asymptotic series for the integral

$$\int_a^\infty \frac{e^{-\lambda x}}{x} dx \tag{2}$$

for large λ. Write a Python script to evaluate the series out to n terms for any given a. Discuss the behavior as a function of n. At what point does the series start to diverge? What is the optimal value of n?

4 Asymptotic series for back radiation

Let $T(p)$ be the temperature of a planet's atmosphere as a function of pressure, and let p_s be the surface pressure (pressure goes down with altitude). The expression for the infrared radiation I_- impinging on the planet's surface is

$$I_- = -\int_{p_s}^0 \sigma T(p)^4 e^{-\kappa(p_s - p)} dp \tag{3}$$

where κ is a constant measuring the absorptivity of the atmosphere. Find a two-term asymptotic series for the surface radiation, in terms of the value of temperature and is derivative at the surface.

5 Asymptotic series for functions that are not infinitely differentiable

Discuss the large λ behavior of the integral

$$\int_0^\infty \frac{e^{-\lambda x}}{1 + \sqrt(x)} dx \tag{4}$$

To how many terms can you take the asymptotic series before the expansion breaks down?

6 Steepest descent and Stirling's approximation to the factorial.

The Gamma function is defined by the integral

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx \tag{5}$$

for any s with a non-negative real part. Using integration by parts successively it is easily shown that $\Gamma(n) = (n-1)!$ when n is a positive integer.

Use the steepest descent method to find the approximate value of $\Gamma(s)$ for large s. Applied to integers, this gives you an approximate formula for the factorials of large numbers, which is known as *Stirling's approximation*.

7 Accuracy of Runge-Kutta for badly behaved functions

The derivation of Runge-Kutta assumes that the function on the right hand side of the ODE has four continuous derivatives. What happens if this condition is not satisfied? To explore this question, numerically integrate the equation

$$\frac{dy}{dx} = 1 - y^{\frac{1}{3}} \tag{6}$$

in the inteval $[0, \frac{3}{2}]$ subject to initial condition with $y(0) = 0$, and compare with the analytic solution. Does the solution converge as the step size is reduced? How rapidly?

For the sake of comparison, implement the second order midpoint method, and re-do the above exercise. Is fourth-order Runge-Kutta superior to the midpoint method for this problem?

A good way to analyze convergence is to make a function `error(dx)` which returns the absolute value of the difference between the computed value at the endpoint and the exact solution there. Then, you can examine `error(dx/2)/error(dx)` for various `dx`, to determine the rate at which the error decreases. Alternately, you can make plots of `log(error(dx))` vs `log(dx)`, and get the order of convergence by measuring the slope. Remember that machine precision for Python floats is about 10^{-14}, so you do not expect further improvements in the solution once the error reaches this threshold.

Compare the convergence for fourth-order Runge-Kutta in the above case with the convergence you get for the well-behaved system

$$\frac{dy}{dx} = -2xy \tag{7}$$

in the interval $[0, 1]$. This equation can also be integrated analytically, which allows you to analyze the error easily.

Another interesting case to consider is

$$\frac{dy}{dx} = \sqrt{1 - y} \tag{8}$$

in the interval $[0, 2]$, subject to $y(0) = 0$. Show that this equation has a quadratic solution, which is in fact infinitely differentiable. Analyze the convergence of fourth-order Runge-Kutta applied to this system, and show that the empirical convergence rate is only second order. Based on the way Runge-Kutta is derived, why do you think the convergence is degraded in this case?

8 Solution basis for constant coefficient systems

(a) Find the solution basis for the "critically damped" harmonic oscillator:

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + y = 0 \tag{9}$$

defined by $b = 2$. Show that you can obtain the same basis by taking suitable sums and differences of the independant exponential solutions for $b > 2$, and then letting b approach 2.

(b) Find the solution basis for the problem

$$\frac{d^4y}{dx^4} - y = 0 \tag{10}$$