Mathematical Modeling and Differential Equations

A quarter abroad in Paris

December 6, 2004

1 Preliminaries

1.1 Overview of the program

This is a syllabus for a three-course sequence making up a study-abroad program to be taught beginning Winter Quarter 2005 at the University of Chicago's Paris Center. In addition to covering a range of topics in introductory dynamical systems, numerical methods, and asymptotic methods, it is intended to cover all the material that would ordinarily be covered by Math 27300, and provide the prerequisite necessary for students to go on to the PDE course, Math 27500. Most of the material from 27300 is covered in Part 2 of the sequence, but some of the more introductory topics from 27300 are done in Part 1. We are anticipating that Math departmental credit will be given for Part 2 of the sequence. It would be desirable (though less essential) for departmental credit in Mathematics to be allowed for Part 1 as well. The nature and rigor of the material covered in Part 1 is similar to that of some of the existing math department listings of a more physics-oriented nature, e.g. 22000,22100 or 29200. However, Part 1 could equally appropriately be given a CS listing. The question of an appropriate physical-science departmental home for Part 3 will be taken up once the syllabus is more complete.

The sequence consists of three courses, each of which will have its own course number and confer credit as a one-quarter University of Chicago course. Each course is divided into an "a" part and a "b" part which will be taught concurrently. This division is partly a matter of separating out different groups of topics, but is also meant to facilitate sharing the teaching with a lecturer to be hired from one of the Paris institutions of higher learning (e.g. Polytechnique or ENS). The assistant lecturer would teach part or all of the "Part b" lecturers. There will also be one graduate student TA, who will run weekly problem and discussion sections.

Each lecture is 1.5 hours in length. For each of the three courses, there will be 4 "Part a" lectures per week and 2 "Part b" lectures per week for 3 weeks.

There will be a final exam at the end of each of the three courses. Weekly problem sets will include a mix of proofs, analytic computations and computer exercises.

1.2 Prerequisites

The minimal prerequisites are Math 15100-15200 or 16100-16200. In addition, the student will need to have some basic familiarity with linear algebra, at least to the extent of being familiar with matrices, determinants, matrix multiplication and inversion and the general notion of an eigenvalue problem.

Computer exercises will be carried out using the Python programming language, but no prior programming experience is presumed.

1.3 Textbooks and Readings

A provisional choice of textbooks is:

- Norm Lebovitz' online differential equation textbook (www.cs.uchicago.edu/ lebovitz)
- Bender and Orszag *Advanced Mathematical Methods for Scientists and Engineers*
- Isaacson and Keller *Analysis of Numerical Methods* (Used in Part 3a).
- Magnus Lee Hetland *Practical Python*

Additional material for the course is drawn from the following texts, which the students will not need to purchase. Selected materials from these works will be made available as e-reserve on the web, and copies will be available in the Paris Center library.

- Birkhoff and Rota *Ordinary Differential Equations.* (referred to as BR in the syllabus)
- Hirsch, Smale and Devaney *Differential Equations, Dynamical Systems and an Introduction to Chaos*, second edition.
- Press, *et al*, *Numerical Recipes*

Additional supplemental readings to be put on e-reserve include:

- Chapter 5 of Laurent Schwartz' *Cours d'Analyse.* (in French)
- Fourier's article on calculation of the subsurface temperature distribution of the Earth (both in French and English).

In addition, we will maintain a small library covering various cultural and historical dimensions of French mathematics and science, e.g. Laurent Schwartz' autobiography.

Part 1: CS 28501 (Topics in Scientific Computing)

Part 1a: Introduction to Dynamical Systems, Numerical Methods and Programming

The emphasis in Part 1a is on defining basic concepts and presenting examples that help provide a foundation for understanding the more general development given in Part 2.

Lecture 1: Discrete dynamical systems I

Introduction. What is a dynamical system. Discrete dynamical systems. 1D maps on an interval of the real numbers. The logistic map

$$x_{n+1} = ax_n(1 - x_n) \tag{1}$$

Motivation of the problem, in terms of problems in population biology.

Fixed points. Linearization and stability. Relation of linear stability to stability of the full nonlinear system.

Lecture 2: Discrete dynamical systems II

Periodic orbits and their stability. Period doubling bifurcations. Chaotic behavior. Some properties of chaos. Sensitive dependence on initial conditions. Lyapunov exponents. Unstable periodic orbits are dense.

Other 1D maps (Tent map. Bernoulli map. Baker's transformation).

Number theory meets dynamical systems: the $3n + 1$ conjecture.

A brief discussion of maps on the plane.

Lectures 3 and 4: Python

Note: We begin introducing Python here, so students will have it available as an aid to calculation on problem sets, and also because it is of use in illustrating examples of asymptotic approximation in the 1b lectures

Basics of programming in Python: Variables, lists,modules,functions, loops, conditionals. Uses of list comprehension and list methods. Using the Python development environment to write and edit programs.

Objects and object-oriented programming;Overloading operators (use $Z[\sqrt{(5)}]$ as example). Programming example: A continued-fraction object.

The `Numeric` array module; Use of the simple plotting package;Writing out your results to a file

Analysis of 1D dynamical systems will be used throughout these lectures to illustrate the use of Python in exploratory numerical work.

Lecture 5 and 6: Introduction to ODEs

Definition of an ordinary differential equation;Differential equations defined in terms of a map on function spaces; Linear vs. nonlinear maps, and why linear is easier (superposition); One should think of solution of O.D.E. as a matter of substituting in a function and see if it maps to zero, not primarily in terms of "sniffing out" the solution from a starting point; ODE's have *families* of solutions, and auxiliary (e.g. initial) conditions must be specified to pick out which one we want (examples of solution families, along the lines of BR Chapter 1).

Solution methods for first order equation initial value problems. Separation of variables. Integrating factors. General solution to the first order linear ODE.

Scientific examples:

- Radiative cooling of an object. $\frac{dT}{dt} = -aT^4$
- Logistic equation (example of growth and saturation). $\frac{dC}{dt} = (1-C)C$
- Finite time blowup. Condorcet's equation. $\frac{dC}{dt} = (1+C)C$. Condorcet's philosophy (the anti-Malthus), and his sad fate.
- Ocean mixed layer, Newtonian cooling, forced by oscillating heat source. Use of complex exponentials in solution.

Comparison theorems, and proof of uniqueness. (Existence proof in Part 2)

Lectures 7 and 8: Linear ODEs

General discussion of linear, second-order differential equations with nonconstant coefficients: linear independence, bases, Wronskian; the variation-of-constants method for the inhomogeneous problem and the influence (or Green's) function; the system formulation; Poincaré phase plane.

Oscillation theorems.

Examples. The Harmonic oscillator. Green's function for the damped harmonic oscillator initial value problem.

Lecture 9: Equations with constant coefficients

Linear differential equations with constant coefficients; the characteristic polynomial; multiple roots; the method of undetermined coefficients for the inhomogeneous problem. The uniqueness theorem.

Lecture 10: Systems of equations

The reformulation of linear equations with constant coefficients as linear systems; system formulation of the general initial-value problem; autonomous, nonlinear systems and the stability of their equilibrium points; "Autonomization" (turn N-d non-autonomous into (N+1)-d autonomous).

Analysis of Foucault's Pendulum. Field trip to the Pantheon to see it.

Lectures 11 and 12: Numerical solutions

Introduction to concept of numerical approximation; reduction from infinite dimensional operator to finite dimensional operator; "Consistency" (convergence; accuracy of approximation; Approximation of first and second derivatives by finite difference (second order). Taylor polynomials for functions that don't have a convergent Taylor series, or even infinitely many derivatives. Methods for numerical solution of ODE. Use of Autonomization. Euler method. Midpoint method (RK2); derivation of fourth-order Runge-Kutta; implementation of Runge-Kutta in Python; error estimates; (Adaptive step-size implementation deferred). Examples of performance; What happens if the rhs is has fewer than four derivatives?

Approach to RK derivation: Derive coefficients for one-step (Euler) and two-step (midpoint) methods. Then turn to 4th order case, which is done by the same idea, but with more algebra. A lot more algebra.

Numerical stability. Illustration for Euler method applied to exponentially damped 1D system.

Part 1b

Lecture 1: The complex plane I

This lecture and the next is intended to provide a basic grounding in the few concepts from complex variables that are used in this course, primarily to serve the needs of those who have not taken a course in the subject. We will not have time to prove any of the major results, but these lectures will at least provide enough background that a certain selective amount of work in the complex plane can be used to enhance some of the material. It is anticipated that students who have not taken complex variables will someday fill in the gaps with a full course in the subject. An elementary treatment of the material can be found in many books, among others Churchil *Complex Variables with Applications.*

The first lecture will cover: Complex numbers, the complex plane, complex differentiation, elementary analytic functions , de Moivre's theorem, basic properties of analytic functions (a once-differentiable function is infinitely differentiable and has a Taylor series with finite radius of convergence); one-dimensional complex dynamical systems – generation of the Mandelbrot set, perhaps some other examples.

Lecture 2: The complex plane II

Integration in the complex plane; real and complex improper integrals and their evaluation; statement of Cauchy's theorem;

Lecture 3: Asymptotic evaluation of integrals I

Integrals of the form

$$\int_a^b e^{-\lambda x} f(x) dx \tag{2}$$

as $\lambda \to \infty$. Development of series through integration by parts. This lecture serves also as an introduction to the general notion of asymptotic approximations, and includes a definition of "asymptotic series." Discuss implication of singularities in derivatives of f, interior to the interval $[a, b]$.

(Based on Bender and Orszag, Chapter 6)

Lecture 4: Asymptotic evaluation of integrals II

The method of steepest descent for asymptot ic evaluation of integrals of the form

$$\int_a^b e^{-\lambda f(x)} dx \tag{3}$$

as $\lambda \to \infty$. Some interesting applications (e.g. Central Limit Theorem) will be discussed in class. Others will be developed in the problem sets

(Bender and Orszag, Chapter 6, but restricted to real-valued f(x))

Lecture 5: Asymptotic evaluation of integrals III

Stationary phase approximation for asymptotic evaluation of integrals of the form

$$\int_a^b e^{-i\omega(k)t} dk \tag{4}$$

as $t \to \infty$, where $\omega(k)$ is real. Application to integrals of the form

$$\int_{-\infty}^{\infty} A(k) e^{i(kx-\omega(k)t)} dk \tag{5}$$

and the notion of phase and group velocities.

(Bender and Orszag, Chapter 6)

Lecture 6: Numerical evaluation of integrals

General approach to approximation as finite sum. Relation to definition of Riemann and Stieltjes integral. Rectangular rule. Proof of convergence assuming that the integrand is continuous and bounded. Trapezoidal rule. Sketch of proof of second order convergence when the integrand is continuously differentiable. Sketch of higher order methods based on trapezoidal rule and extrapolation.

(Based on *Numerical Recipes*, Chapter 4)

Part 2: Math 27301 (Ordinary Differential Equations

Part 2a: Theory of Ordinary Differential Equations

Lectures 1-4: Existence and regularity for the initial-value problem.

Follows sections 6.1 - 6.4 and 6.6-6.9 of BR

Existence and uniqueness for vector initial value problem on R, i.e.

$$\frac{d\mathbf{Y}}{dx} = \mathbf{F}(Y, x), \mathbf{Y}(0) = \mathbf{Y}_0 \tag{6}$$

Reduction to an integral equation; review of convergence theorems; Picard's theorem; continuation theorem; fixed-point methods; uniqueness; smoothness of dependence of solution on initial data and on parameters. Global existence/uniqueness for linear equations is a corollary.

Lectures 5 and 6: Linear equations in the complex plane

Series solutions of second-order linear equations in the neighborhood of a point where all coefficients are analytic; radius of convergence; solutions of equations in the neighborhood of singular points; definition of 'regular singular point.' method of Frobenius for solutions near regular singular points. Examples will be discussed in Part b.

Lectures 7 and 8: Linear boundary-value problems

Examples of boundary-value problems from classical and quantum physics ; operators and their adjoints; self-adjointness: reality of eigenvalues and orthogonality of eigenfunctions; Green's function for the solution of the inhomogeneous problem; solvability condition.

Lectures 9 and 10: the Sturm-Liouville boundary-value problem

Definition of Sturm-Liouville systems; Prufer substitution; oscillation and comparison theorems; the sequence of eigenfunctions. The distribution of eigenvalues and asymptotic form of the eigenfunctions, if time permits.

Lectures 11 and 12: Eigenfunction expansions

Ideas of convergence for series of functions; Rayleigh-Ritz characterization of eigenfunctions and eigenvalues; expansion of arbitrary functions in series of eigenfunctions of the Sturm-Liouville problem.

Part 2b

Lecture 1: Planar, autonomous systems I

RTP will give this lecture and the next

Stability of equilibrium points: classification of types of behavior and illustration with orbits of simple Hamiltonian systems and limit cycles of simple dissipative systems. Statement of Poincare-Bendixson theorem and beginning of proof.

Lecture 2: Planar, autonomous systems II

Complete discussion of Poincare-Bendixon theorem.

Lecture 3: The Lorentz system

This is and the following lecture constitute a "special topic,"

Definition of the system. Fixed points and their stability. Numerical exploration of the system. The Poincare section for this attractor.

(There are many good references for this, but I especially like Chapter 14 of Hirsch, Smale and Devaney).

Lecture 4: Orbital mechanics

The two body problem for Newtonian gravity. Solution using conservation of angular momentum and energy. Orbits are planar and elliptical. (Any good physics textbook does this problem, but see also Chapter 13 of Hirsch, Smale and DeVaney).

The restricted 3 body problem (i.e. motion of an almost massless planet in the gravitational field of a pair of massive co-orbiting planets). Resonance. Chaotic orbits. Illustration of behavior using numerical integration.

(I don't know of a good general reference to this. There may not be time in the lecture for it anyway. In any event, the idea is just to point out the problem posed by resonance between the orbital periods, and show some numerical solutions illustrating the chaotic behavior. This is the problem that led Poincaré to the discovery of the field of dynamical systems.)

Lectures 5 and 6: Special functions and singular boundary-value problems

The (non Sturm-Liouville) eigenfunction problems for certain special functions of mathematical physics, and their completeness: Legendre functions; Bessel functions; Hermite polynomials. (This material can be found in Lebovitz' online book, and also in Birkhoff and Rota.)

Part 3: Math 211, or CS Reading and Research (for those who have had 211 already)

Part 3a

This section will be taught by Ridg Scott. It is an enhanced and more advanced version of the syllabus usual for 211, which will take into account the greater preparation students will have had by the time they reach this part. It is based on the text by Isaacson and Keller.

Part 3b

Lecture 1: WKB

The WKB approximation. Application to analysis of asymptotic properties of Sturm-Liouville problems.

Lecture 2: Numerical solution of ODE boundary value problems

Reduction of second order boundary value problem to a tridiagonal linear system, by centered differences; implementation of boundary conditions; Implementation in Python, and examples of solutions (in the exercises).

Discussion of problem with "shooting method" (i.e. direct use of resolvent), regarding exponentially growing solutions.

(Can be based on Chapter 17 of *Numerical Recipes*, restricted to the second order linear case)

Lecture 3,4: Nonlinear oscillators and chaos

Analysis of the anharmonic oscillator using perturbation theory. (Bender and Orszag, pp 544-551)

Analysis of chaos in periodically driven nonlinear oscillators. The periodically forced Duffing's equation (anharmonic oscillator).

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = A\cos(\omega t) \tag{7}$$

Numerical exploration. Use of Poincaré sections. Nonlinear detuning of resonance. Stable and unstable manifolds. Numerical investigations.

A few remarks on the physical realization of this system as a pendulum on an oscillating fulcrum. Possible field trip to La Villete to play with their double-pendulum.

(This could be based on the introductory chapters of Holmes and Guckenheimer, *Nonlinear Oscillators....* That treatment is a little terse and difficult, but I don't really have a good alternate reference for this).

Lecture 5,6: Special topics

Free choice of subject, for the guest lecturer to talk about his or her current research interests.