Geosci 342 Problem Set 2

February 13, 2009

Errata Equation (27) of Problem Set 1 solution has units of radians per unit time.

Problem 2.1 (a) The acceleration due to the centrifugal force is $\Omega^2 r$, where r is the distance from the axis of rotation. At the equator, r is simply the Earth's radius. Compare the centrifugal acceleration at the Equator with the acceleration of gravity on Earth...

At the equator

$$\Omega^2 r \approx \left(\frac{2\pi}{24 \cdot 3600 s}\right)^2 \cdot (6370 \cdot 10^3 m)$$

so the centrifugal acceleration is

$$\Omega^2 r \approx 3.4 \cdot 10^{-2} m\ s^{-2}$$

which is much less than gravity. At 45° latitude

$$r \approx \cos(\pi/4) \cdot (6370 \cdot 10^3 m)$$

so the centrifugal correction at 45° is

$$\frac{1}{2}\rho\Omega^2 r^2 \approx \frac{1}{2}(1\ kg\ m^{-3})\left(\frac{2\pi}{24 \cdot 3600 s}\right)^2 \cdot \Big(\cos(\pi/4) \cdot (6370 \cdot 10^3 m)\Big)^2$$

$$\approx 5.365 \cdot 10^4\ kg\ m\ s^{-2}\ m^{-2} = 5.365 \cdot 10^4\ N\ m^{-2} = 5.365 \cdot 10^4\ Pa$$

At 45° latitude and 10 km above the surface

$$r \approx \cos(\pi/4) \cdot (6380 \cdot 10^3 m)$$

so the centrifugal correction at 45° and 10 km above the surface is

$$\frac{1}{2}\rho\Omega^2 r^2 \approx \frac{1}{2}(1\ kg\ m^{-3})\Big(\frac{2\pi}{24\cdot 3600s}\Big)^2 \cdot \Big(\cos{(\pi/4)}\cdot(6380\cdot 10^3 m)\Big)^2$$
$$\approx 5.382\cdot 10^4\ Pa$$

The difference between centrifugal correction at 10 km and the surface is about $1.69\cdot 10^2 Pa$ whereas the surface pressure is $10^5 Pa$.

Problem 2.2 In the absence of a pressure gradient, the velocity of a fluid parcel measured in a rotating reference frame obeys the equation

$$\frac{d}{dt}\vec{v} = -2\Omega\hat{z}\times\vec{v} \tag{1}$$

where $\vec{v} = (u, v)$ is the horizontal velocity vector. Note that this can be treated as an ordinary differential equation if you are following along with the fluid parcel, because there are no pressure gradients to couple one fluid parcel to another. Integrate the differential equation to show that, following a fluid parcel, the magnitude of the velocity vector remains constant and that its direction rotates uniformly over time; find the rotation rate.

Writing equation (1) in component form

$$\frac{du}{dt} = 2\Omega v \tag{2}$$

$$\frac{dv}{dt} = -2\Omega u \tag{3}$$

We want to solve this system of equations. One way to do this is by multiplying the first equation by u and the second equation by v,

$$u\frac{du}{dt} = 2\Omega vu \tag{4}$$

$$v\frac{dv}{dt} = -2\Omega uv \tag{5}$$

Using the product rule (in reverse) these equations become

$$\frac{1}{2}\frac{du^2}{dt} = 2\Omega vu \tag{6}$$

$$\frac{1}{2}\frac{dv^2}{dt} = -2\Omega uv \tag{7}$$

Adding the two equations together,

$$\frac{du^2}{dt} + \frac{dv^2}{dt} = 0 \tag{8}$$

this is an exact differential that readily leads to an equation for the magnitude of the velocity vector

$$u^2 + v^2 = A^2 \tag{9}$$

which means the velocity vector describes circular motion,

$$u = A\cos\theta \tag{10}$$

$$v = A\sin\theta \tag{11}$$

Substituting equations (10,11) into equation (6)

$$\frac{d}{dt}(\cos^2\theta) = 2\Omega\cos\theta\sin\theta \tag{12}$$

this becomes

$$-2(\cos\theta\sin\theta\frac{d\theta}{dt}) = 2\Omega\cos\theta\sin\theta \tag{13}$$

so the rotation rate is $-\Omega$ (which means the flow is anticyclonic and moves around half of one circle in one Earth day).

Problem 2.3 For a planar velocity field (u, v) the vertical vorticity is $\partial_x v - \partial_y u$. Suppose we transform to a rotated coordinate system (x', y') in which the velocity field has the components (u', v'). Show that in the rotating frame the vorticity is $\partial_{x'} v' - \partial_{y'} u'$, i.e. the same form as it had in the rest frame.

There are several approaches to this problem. The following argument aims to cover the concept of a 2D reference frame rotating with an angular velocity constant in time. Consider the position vector in the fixed (non-rotating, or inertial) reference frame,

$$\vec{r} = x\hat{x} + y\hat{y} \tag{14}$$

Taking the material derivative and using the chain rule,

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + x\frac{d\hat{x}}{dt} + \frac{dy}{dt}\hat{y} + y\frac{d\hat{y}}{dt} \tag{15}$$

For a coordinate system rotating at a constant rate (Ω), the unit vectors ($\hat{x}$ and $\hat{y}$) sweep out differential arc lengths in directions perpendicular to the unit vectors

$$\frac{d\hat{x}}{dt} = ||\hat{x}||\frac{d\theta}{dt}\hat{y} = \Omega\hat{y} \tag{16}$$

$$\frac{d\hat{y}}{dt} = -||\hat{y}||\frac{d\theta}{dt}\hat{x} = -\Omega\hat{x} \tag{17}$$

We can write the fixed-reference frame velocity vector (left hand side of equation 15) in terms of the velocity in a rotating reference frame by taking the rotation rate (Ω) to be nonzero. When we do this the dx/dt and dy/dt terms on the right hand side of equation (15) become the relative velocities (u'_{rel} and v'_{rel}); primes indicate that we are in a reference frame rotated relative to the fixed reference frame; and the subscripts (rel) indicate that this reference frame is also undergoing continual rotation over time so that the velocities are measured relative to the motion of the rotating reference frame. Using these relations, equation (15) becomes

$$u' = u'_{rel} - y'\Omega \tag{18}$$

$$v' = v'_{rel} + x'\Omega \tag{19}$$

which states the absolute velocity in the fixed, rotated (but not rotating) reference frame (u', v') is given by the relative velocity in the rotating reference frame plus an additional factor due to the rotation rate of the reference frame. The vorticity is obtained from differentiating equation (18, 19),

$$\frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} = \frac{\partial v'_{rel}}{\partial x'} - \frac{\partial u'_{rel}}{\partial y'} + 2\Omega \tag{20}$$

The left hand side (LHS) is the absolute vorticity (the vorticity in the non-rotating frame), which is equal to relative vorticity in the rotating reference frame (first two RHS terms) plus twice the rotation rate of the reference frame.

We can show that rotation of the coordinate system does not change the absolute vorticity by using the chain rule

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial v}{\partial y'}\frac{\partial y'}{\partial x} \tag{21}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} \tag{22}$$

By differentiating the rotation matrix defined in class we can rewrite equations (21, 22) as

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x'}\cos(\Omega t) - \frac{\partial v}{\partial y'}\sin(\Omega t) \tag{23}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y'}\cos(\Omega t) + \frac{\partial u}{\partial x'}\sin(\Omega t) \tag{24}$$

Forming the vorticity from these equations we obtain

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x'}\Big(v\cos(\Omega t) - u\sin(\Omega t)\Big) - \frac{\partial}{\partial y'}\Big(v\sin(\Omega t) + u\cos(\Omega t)\Big) \tag{25}$$

The RHS of equation (25) can be rewritten in terms of the velocity components in the rotated reference frame, to obtain

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} \tag{26}$$

which means the absolute vorticity is unchanged by rotation of the coordinate axes. We can now rewrite equation (20)

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial v'_{rel}}{\partial x'} - \frac{\partial u'_{rel}}{\partial y'} + 2\Omega \tag{27}$$

which shows that the absolute vorticity in the fixed reference frame (LHS terms) is equal to the relative vorticity in the rotating frame (first two RHS terms) plus an additional constant 2Ω due to the rotation rate of the coordinate system (e.g. the Earth's vorticity).

Problem 2.4 The vertical vorticity equation in the rest frame (i.e. the non-rotating frame) is

$$\frac{d}{dt}\omega_s - \vec{\omega}\cdot\nabla w = 0 \tag{28}$$

We derive the vertical vorticity equation in the rotating reference frame by splitting the vertical vorticity into the Earth's part and the relative part and substituting into equation (28) to get

$$\frac{d}{dt}\Big(2\Omega\hat{z} + \omega_{z,rel}\Big) - \vec{\omega}\cdot\nabla w = 0 \tag{29}$$

There is no time rate of change of the Earth's vorticity and the advection term in the material derivative is invariant to rotation of the coordinate axes (per discussion in class). Furthermore the results from problem 2.3 showed that the absolute vorticity in a rest frame is equal to the sum of the Earth's vorticity and the relative vorticity in the rotating coordinate system (see equation 27). Considering these facts, the first LHS term of equation (29) becomes the time rate of change of the relative vorticity in a reference frame rotating with the Earth's vorticity.

$$\frac{d}{dt}\omega'_{z,rel} \tag{30}$$

For the second LHS term, note that dot product in the full vorticity equation derived in class ($\vec{\omega}\cdot\nabla\vec{v}$) only contributes the term ($\vec{\omega}\cdot\nabla w$) to the z-component of vorticity (ω_z). Expanding this term gives,

$$\vec{\omega}\cdot\nabla w = \omega_x\frac{\partial w}{\partial x} + \omega_y\frac{\partial w}{\partial y} + \omega_z\frac{\partial w}{\partial z} \tag{31}$$

$$\vec{\omega}\cdot\nabla w = \Big(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\Big)\frac{\partial w}{\partial x} - \Big(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\Big)\frac{\partial w}{\partial y} + \Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)\frac{\partial w}{\partial z} \tag{32}$$

Taking the horizontal velocity independent of height, equation (32) becomes

$$\vec{\omega}\cdot\nabla w = \frac{\partial w}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)\frac{\partial w}{\partial z} \tag{33}$$

The first two terms cancel, leaving

$$\vec{\omega}\cdot\nabla w = \Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)\frac{\partial w}{\partial z} \tag{34}$$

The first RHS quantity is the absolute vorticity in the rest frame, which can be written as the absolute vorticity in the rotating reference frame using the result of equation (26). Using these results, (30), and the invariance of the dot product to coordinate rotation, the relative vorticity equation becomes

$$\frac{d}{dt}\omega'_{z,rel} - \Big(\omega'_{z,rel} + 2\Omega\Big)\cdot\nabla' w = 0 \tag{35}$$

Problem 2.5 Consider a region of the Earth where $2\Omega \approx 10^{-4}s^{-1}$. A uniform geostrophically balanced wind with a speed of 20 $m\ s^{-1}$ blows through this region. The air density is 1 $kg\ m^{-3}$. How much does the pressure in this region vary over a distance of 1000 km? Give the answer in Pascals. Do the same for an ocean current of 10 $cm\ s^{-1}$ with density of water 1000 $kg\ m^{-3}$

A geostrophically balanced wind in (suppose) the x-direction is given by the x-component of the equation for geostrophic balance

$$2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

so the pressure gradient is given by

$$-2\rho\Omega u = \frac{\partial p}{\partial y}$$

which gives a pressure gradient of magnitude 0.002 $Pa\ m^{-1}$. Over 1000 km the pressure will vary by $2 \cdot 10^3$ Pa, or 2 mb. For the ocean the pressure varies by $1 \cdot 10^4 Pa$.

Problem 2.6 Find the geostrophically balanced flow over a ridge with height $h(x) = h_0(1 - x^2/L^2)$ for $|x| < L$, having $h = 0$ elsewhere. The fluid layer has depth D away from the ridge, and the flow far upstream is in a uniform geostrophically balanced current in the x direction with constant speed U. Sketch the vorticity field, marking regions of positive and negative vorticity. Interpret your result in terms of the stretching or compression of the Earth's vorticity of rotation.

Starting with the vertical vorticity equation (first order in Rossby number) derived in class,

$$\frac{d_g\omega}{dt} = -\frac{2\Omega}{D}\frac{dh(x)}{dt} \tag{36}$$

where the RHS is proportional to the change in bottom height, assuming the change in surface height to be relatively small. This states that the sum of the relative vorticity and a term proportional to the bottom height is constant following the fluid motion,

$$\frac{d_g}{dt}\left(\omega + \frac{2\Omega}{D}h(x)\right) = 0 \tag{37}$$

so that

$$\omega + \frac{2\Omega}{D}h(x) = 0 \tag{38}$$

where we take the constant to be zero because the relative vorticity must vanish far from the ridge if the flow there has constant speed U. Writing this in terms of streamfunction and moving the height term back to the RHS,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\frac{2\Omega h(x)}{D} \tag{39}$$

We know that in the far stream $\partial_y \psi$ is constant in y because U is constant. Since the ridge height is also constant in y we have $\partial_y \psi' = 0$ for the perturbation near the ridge. Imposing these requirements on equation (39) gives a perturbation streamfunction equation valid near the ridge

$$\frac{d^2 \psi'}{dx^2} = -\frac{2\Omega h(x)}{D} \tag{40}$$

Integrating once and taking the constant of integration to be zero (for symmetry of the perturbation y-velocity about the ridge)

$$\frac{d\psi'}{dx} = \frac{2\Omega h_0}{D}\Big(- x + \frac{x^3}{3L^2}\Big) \tag{41}$$

Integrating a second time,

$$\psi' = \frac{2\Omega h_0}{D}\Big(- \frac{x^2}{2} + \frac{x^4}{12L^2}\Big) + c \tag{42}$$

which gives a solution having high pressure (negative vorticity) over the ridge with surrounding lower pressure (positive vorticity). As the flow moves over the ridge columns of vorticity become compressed. Since the change in column depth is negative for compression the relative vorticity must decrease over the ridge to satisfy potential vorticity conservation (equation 36). Since the flow is geostrophic we know the perturbation y-component of the velocity (v) is positive (northward) to the left of the ridge and negative (southward) to the right of the ridge to give an anticyclonic (negative vorticity) circulation.