Geos 232 2009 HW2 Solutions

Problem 2.1

The energy per unit frequency per unit area, emitted in all directions leaving the surface is $\pi B(\nu, T)$. The factor of π comes from integrating emission over all angles. Hence, the net energy in a frequency band of width $\Delta\nu$ leaving the surface of a body with surface area A, per unit time, is $\pi B(\nu, T)A\Delta\nu$. We need to convert the wavenumber band to a frequency band, which is easily done since $\nu = c \cdot n$, where n is the wavenumber. If we want to use the speed of light, c in m/s (rather than cm/sec), then we also need to convert the wavenumbers to inverse-meters, which we do by multiplying by $100 cm/m$. Hence, $\Delta\nu = 100. \cdot (750 - 500) \cdot c = 7.5 \cdot 10^{12} Hz$ Similarly, the frequency at the center of the band is $\nu = 1.9 \cdot 10^{13} Hz$. The area is $A = 4\pi \cdot (1m)^2$. Plugging in, we find $\pi B(\nu, 300)A\Delta\nu = 1507W$. The total emission over all wavenumbers is $\sigma \cdot 300^4 A = 5771W$. The stated wavenumber band contains about a quarter of the total emission. Note that the assumption that B is nearly constant over the wavenumber range is quite accurate. For example, a direct calculation shows $B(\nu + \Delta\nu/2., 300.)/B(\nu, 300.) = 1.08$, so B is constant to within about 8%.

Problem 2.2

The wavenumber of maximum emission is $k_{max} = \nu_{max}/c \simeq 2.821\frac{k}{ch} \cdot 200K \simeq 392.14 cm^{-1}$. The rate of energy emission from a $1m^2$ patch, over the $100 cm^{-1}$ band centered on k_{max} is given by $F(T) = 1m^2 \int_{k_{max}-50}^{k_{max}+50} \pi B(c \cdot k, T) d(c \cdot k)$. If we assume that the Planck function is constant over this wavenumber range the inegral can be approximated by $F \simeq 1m * \pi * B(n_{max}, 200) * (c * 10,000 m^{-1}) = 14.3W$. To assess this assumption we can calculate $B(n_{max} + 50cm^{-1}, 200)/B(n_{max} - 50cm^{-1}, 200) \simeq 1.003$ and find that the Planck function varies less than 1% over this interval.

Problem 2.3

If the flux from a star a frequency ν_j is approxamitely: $F_j = \pi B(\nu_j, T)\frac{R_\odot^2}{r^2}$ where $R_\odot$ is the radius of the star and and r the distance to the observer, then the astronomical magnitude $M_j = C_j - 2.5logF_j = C_j - 2.5(log(\pi B(\nu_j, T)) + 2log(\frac{R_\odot}{r}))$. Then the color index $M_2 - M_1 = C_2 - C_1 - 2.5log(\frac{B(\nu_2,T)}{B(\nu_1,T)})$ which is independent of distance. In the classical limit $h\nu \ll kT$ so $B(\nu, T) \simeq \frac{2\nu^2 k}{c^2}T$ so the color index becomes independent of temperature $M_2 - M_1 = C_2 - C_1 - 5log(\frac{\nu_2}{\nu_1})$. In the quantum limit $h\nu \gg kT$ so $B(\nu, T) \simeq \frac{2h\nu^3}{c^2}e^{-\frac{h\nu}{kT}}$ and the color index is $M_2 - M_1 = C_2 - C_1 - 7.5log(\frac{\nu_2}{\nu_1}) + \frac{2.5hlog(e)}{kT}(\nu_2 - \nu_1)$. Solving for temperature we find an expression the photospheric temperature of a star in terms of its color index $T = 2.5hlog(e)(\nu_2 - \nu_1)/(M_2 - M_1 - C_2 + C_1 + 7.5log(\nu_2/\nu_1))k$. The common B-V color index employs the "Blue" band at $.44\mu m$ and the "Visual" band at $.55\mu m$. Comparing our parametrization to the empirical relation $M_B - M_V = -3.684logT + 14.551$ we find good agreement between 3000K and

7000K if $C_B - C_V \simeq .166$ which gives exact agreement at 6000K.

Problem 2.4

The energy balance with heating from solar absorption alone is

$$2 \cdot r \cdot hL_\odot = (2\pi r^2 + 2\pi r \cdot h)\sigma T^4 \tag{1}$$

so

$$T = [\frac{L_\odot}{\pi(1+\frac{r}{h})\sigma}]^{\frac{1}{4}} \tag{2}$$

Putting in the numbers we get $249K$ (rather chilly). With an extra megawatt dumped into the energy budget, the energy balance becomes

$$2 \cdot r \cdot hL_\odot + 10^6 = (2\pi r^2 + 2\pi r \cdot h)\sigma T^4 \tag{3}$$

Now, we need to make an assumption about the size of the station to get a number. If we still assume $r = h$ but now pick $r = 50m$, then the new temperature is $265K$. The inhabitants might actually like this temperature more than the previous one, but let's assume they're truly arctic creatures and want to get the temperature back down to what it was before. Since we want the temperature to be $249K$, and the radiator will have the same temperature, the radiation from the plate will be $2 \cdot \sigma 249^4 A$, where A is the area of the plate. The factor of 2 is because the plate has two sides. To make this radiate a megawatt, we need $A = 2300m^2$, or roughly 50 meters by 50 meters.

Problem 2.5

If the planet begins to melt at T=273.15K then solar constant necessary a) for an isothermal planet$L = 4\sigma T^4/(1-.7) = 4209W/m^2$, b) if only the sun facing side is in radiative equilibrium $L = 2\sigma T^4/(1-.7) = 2104W/m^2$, c) at the subsolar point with no atmosphere$L = \sigma T^4/(1-.7) = 1052W/m^2$.

Problem 2.6

Given the solar luminosity $L_G = .013L_\odot$ and orbital radii $R_c = .073A.U.$ and $R_d = .25A.U.$ then the equilibrium temperatures are $T_c = (.013L_\odot(1-.3)/.073^2 * 4\sigma)^{.25} = 318.6K$ and $T_d = (.013L_\odot(1-.3)/.25^2 * 4\sigma)^{.25} = 172.1K$. The median wavenumber is a function of temperature $k^{med} = \nu^{med}/c = 3.503kT/ch = 243.47T$. For Gleise and this pair of planets the median wavenumber are: $k_G^{med} = 243.47 * 3480K = 8472.8cm^{-1}$, $k_c^{med} = 775.7cm^{-1}$ and $k_d^{med} = 419cm^{-1}$ and the differences $\Delta k_c^{med} = 7697cm^{-1}$ and $\Delta k_d^{med} = 8300.7cm^{-1}$. Compare this to Mercury whose difference in median wavenumber from our own Sun is $\Delta k_M^{med} = 13050.1cm^{-1}$. Given the temperature and luminosity we can estimate the radius of the star since $L = 4\pi R^2 \sigma T^4$ for Gleise $R_G \simeq (.013 * 3.939 \times 10^{26}W/4\pi\sigma 3480^4)^{.5} = 2.2 \times 10^8 m$.