Supplementary Information for

Earth’s outgoing longwave radiation linear due to H$_2$O greenhouse effect

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This PDF file includes:
Supplementary text
Figs. S1 to S7
References for SI reference citations
Supporting Information Text

1. Validation of line-by-line code.

To validate our code we compare it against the state-of-the-art radiative calculations presented by Goldblatt et al (1). Based on previous studies we consider OLR differences of up to 10 W m$^{-2}$ acceptable. For example, Ramirez et al calculated the runaway greenhouse limit with a modern correlated-k code, and found OLR differences of about 3 – 5 W m$^{-2}$ relative to Goldblatt et al (2). Similarly, Yang et al performed an intercomparison of radiative transfer codes, and found that OLR differed by up to $\sim$7 W m$^{-2}$ between two line-by-line codes which both used modern spectroscopic databases (3).

We performed sensitivity tests which further support our expectation that relatively minor assumptions can affect OLR calculations by several W m$^{-2}$. In particular, we find that OLR is sensitive to the averaged zenith angle used in the longwave two-stream equations. There is no unique choice for this angle (4), and it is often chosen to be $\theta = 53^\circ$ for Earth-like conditions. However, in the hot steam-atmosphere limit radiation should become increasingly isotropic over large parts of the atmospheric column so that the averaged zenith angle should tend towards $\theta = 60^\circ$, at least close to the surface. Alternatively, in optically thin atmospheres the longwave flux could become more upwards peaked, thereby reducing the average zenith angle below $\theta = 53^\circ$. Figure S1 shows that changing the zenith angle from $\theta = 53^\circ$ to $\theta = 60^\circ$ or $\theta = 48^\circ$ can change OLR by up to 6 W m$^{-2}$. We note that more accurate solutions than the two-stream equations exist, so we additionally compared our calculations against a state-of-the-art discrete ordinate solver, DISORT (5), which we adapted using publicly-available software (https://github.com/chanGimeno/pyDISORT). DISORT is numerically expensive to run, and furthermore seems to require very high vertical resolution to run stably at high temperatures, so we restricted ourselves to surface temperatures below 300 K, used 100 vertical grid points, and a vertical top pressure of 0.01 bar. The red crosses in Figure S1 show that OLR calculated with DISORT (we use 16 streams) deviates less than 1 W m$^{-2}$ from our two-stream results with $\theta = 53^\circ$. In the rest of this paper we therefore use an averaged zenith angle of $\theta = 53^\circ$.

Figure S2 shows that our code reproduces the OLR calculations from Goldblatt et al to within 1.5 W m$^{-2}$ for a pure-H$_2$O atmosphere and to within 6 W m$^{-2}$ for an atmosphere with 1 bar of N$_2$ (assumed to be inactive in the infrared). We note that Figure S2 assumes the same vertical temperature-pressure profiles as in Goldblatt et al, to highlight the impacts of different radiative solvers and opacity calculations. If we instead use temperature-pressure profiles calculated with our own code we find OLR differences of up to 7 W m$^{-2}$ in the pure H$_2$O limit, and up to 8 W m$^{-2}$ in the Earthlike limit. The sensitivity to the vertical profiles arises because we use the ideal gas law whereas Goldblatt et al use steam tables for pure H$_2$O atmospheres, and presumably slightly different thermodynamic constants for H$_2$O atmospheres with a dry background gas. In the pure H$_2$O limit the assumed equation of state is therefore important for OLR calculations, but it is of secondary importance for the range of temperatures we focus on in this paper (up to 350 K for H$_2$O atmospheres).

We suspect that the remaining differences between our calculations and those of Goldblatt et al are primarily related to the assumed water vapor continuum. Our code uses the H$_2$O self and foreign continuum opacities calculated by the most recent version of the MTCKD model, version 3.2. In comparison, Goldblatt et al state that their continuum is implemented using $\chi$-factors and “performs similarly in tests to the empirical MT-CKD 2.4 continuum” but “slightly underestimates the strength” of the continuum relative to laboratory measurements (1). Given that our code produces consistently lower OLR than the results in Goldblatt et al (Fig. S2), we believe the remaining differences are therefore due to stronger continuum absorption in the newest version of MTCKD. For comparison, running our code without any continuum absorption leads to an OLR that is more than 100 W m$^{-2}$ higher than the values stated in Goldblatt et al.

Finally, we performed a sensitivity test for the continuum used in our calculations. Recent laboratory measurements by Baranov et al (6, 7) at temperatures relevant for the runaway greenhouse (310 K-360 K) found that continuum absorption could be stronger by up to a factor of two compared with the values used in the MTCKD model at around 1000 cm$^{-1}$, with even larger discrepancies beyond 2000 cm$^{-1}$. The top row in Figure S3 shows the discrepancy between absorption coefficients we calculated using the laboratory data and the MTCKD model. Consistent with Baranov & Lafferty (7), we find that the discrepancy can amount to almost an order of magnitude beyond 2000 cm$^{-1}$. To place an upper bound on the strength of the H$_2$O continuum we therefore formulated a hybrid continuum model, in which we set the absorption cross-section equal to the maximum value given by the MTCKD model and an interpolation of the Baranov et al data, $\kappa(\nu,T) = \max[\kappa_{MTCKD}(\nu,T), \kappa_{Baranov}(\nu,T)]$. The bottom row in Figure S3 shows that the increased absorption in our hybrid model relative to the default MTCKD model decreases OLR by less than 1 W m$^{-2}$, and thus only has a minor impact on the runaway threshold.

2. Limiting expressions for water vapor path.

The inset in Figure 3 shows that the atmospheric water vapor path is an approximately single-valued function of atmospheric temperature over a wide range of surface temperatures. Here we demonstrate why this behavior arises, and further show that the water vapor path at a given temperature is limited to vary by, at most, an order of magnitude across widely different climates.

To place an upper limit on the water vapor path, we consider a pure steam atmosphere. In this limit the water vapor path
is simply

\[ WVP = \frac{1}{g} \int q^*dp \]

\[ = \frac{1}{g} \int dp \]

\[ = e^*(T) \frac{\Gamma(1)}{g}. \]  \hspace{1cm} [1]

Here \( g \) is the acceleration of gravity, \( p \) is pressure, \( q^* \) is the saturation specific humidity (equal to one in the steam limit), and \( e^* \) is the saturation vapor pressure.

To place a lower limit on the water vapor path, we consider a “dry” atmosphere which contains water vapor but in which the water vapor has no significant effect on the atmospheric temperature structure. In this limit the lapse rate is equal to the dry adiabatic lapse rate \( dT/dp = T R_d/(pc_p) \), and the water vapor path is

\[ WVP = \frac{1}{g} \int q^*dp \]

\[ \approx \frac{1}{g} \frac{R_d}{R_v} \int \frac{e^*(T)}{p} dp \]

\[ = \frac{1}{g} \frac{c_p}{R_v} \int_0^{T_{trop}} \frac{e^*(T')}{T'} dT' \]

\[ \approx \frac{1}{g} \frac{c_p}{R_v} \left[ \frac{e^*(T)}{g} \exp\left( \frac{L}{R_v T} \right) \right] \int_0^{T} \frac{R_v T}{L} \]

\[ \approx \frac{c_p T}{R_v g} \frac{e^*(T)}{L}. \]  \hspace{1cm} [2]

Here \( R_d \) is the specific gas constant of dry air, \( c_p \) is the specific heat of dry air, \( R_v \) is the specific gas constant of water vapor, \( L \) is the latent heat, \( \Gamma \) is the incomplete gamma function, and we use the Clausius-Clapeyron relation which states \( e^*(T) = e^*(T_0) \exp(-L/R_v(1/T - 1/T_0)) \). Consistent with our use of the dry adiabat, we assume in the second step that water vapor makes a negligible contribution to atmospheric pressure, in the fourth step that the saturation vapor pressure is much smaller at the tropopause than at the temperature under consideration, and in the sixth step that \( L/(R_v T) \gg 1 \). The last assumption is valid for all condensable substances we consider in this paper over a wide range of temperatures. We note that this “dry” limit was already pointed out by Nakajima et al (8) in the context of the runaway greenhouse, but they did not further simply the full integral form of Equation 2.

The comparison between Equations 1 and 2 shows that the water vapor path in both limits is a single-valued function of atmospheric temperature. Moreover, the maximum possible variation in water vapor path at a given temperature is small and given by the ratio of the two equations, \( c_p T/L \). Here \( c_p \) is a property of the dry background gas while \( L \) depends on the condensing species. For Earth-like conditions with H\(_2\)O condensing in N\(_2\) this value is equal to 0.1, and it is of similar magnitude for a wide range of other atmospheres such as a hypothetical cold early Mars (CO\(_2\)-N\(_2\), 0.6), present-day Titan (CH\(_4\)-N\(_2\), 0.2), early Earth (H\(_2\)O-CO\(_2\), 0.1), or an ammonia-rich world (NH\(_3\)-N\(_2\), 0.1). Our result for H\(_2\)O shown in Figure 3 therefore also applies to atmospheres with other condensable substances.

3. Derivation of simple feedback model.

We start with the equation for the upwards infrared flux of frequency \( \nu \), \( F^+_{\nu} \), at the top-of-atmosphere (4, Chapter 4),

\[ F^+_{\nu} = \pi B_{\nu}(T_0) e^{-\tau_0} + \int_0^{\tau_0} \pi B_{\nu}(T(\tau)) e^{-\tau} d\tau, \]  \hspace{1cm} [3]

where \( B_{\nu} \) is the Planck function, \( T \) is the atmospheric temperature, \( T_0 \) is the surface temperature, and \( \tau \) is the optical depth at the frequency \( \nu \), which varies from \( \tau = 0 \) at the top-of-atmosphere to \( \tau = \tau_0 \) at the surface. The first term is the flux contributed by the surface, and the second term is the flux contributed by the atmosphere. We rewrite this equation using the transmission function, \( T_\nu(\tau) = e^{-\tau} \), so

\[ F^+_{\nu} = \pi B_{\nu}(T_0) T_\nu(\tau_0) + \int_{\tau_0(T_\nu)}^{1} \pi B_{\nu}(T(T_\nu)) dT_\nu. \]  \hspace{1cm} [4]
The contribution of this frequency to the net longwave feedback is given by the derivative of the above equation with respect to surface temperature. Using Leibniz’s rule in the second step, which states \( d\int[a(x)] f(x,y)dy = f(x,b(x)) \, db/dx - f(x,a(x)) \, da/dx + \int[a(x)] df/dx \, dy \), and assuming that the temperature of the air immediately above the surface \( T(\tau_0) \) is equal to the surface temperature \( T_s \), we find

\[
\frac{dF_+^\rho}{dT_s} = \pi \frac{dB_v(T_s)}{dT_s} T_v(\tau_0) + \pi B_v(T_s) \frac{dT_v(\tau_0)}{dT_s} + \frac{d}{dT_s} \left[ \int_{T_v(\tau_0)}^{1} \pi B_v(T(\tau_v)) dT_v \right],
\]

\[
= \pi \frac{dB_v(T_s)}{dT_s} T_v(\tau_0) + \pi B_v(T_s) \frac{dT_v(\tau_0)}{dT_s} - \pi B_v(T(\tau_0)) \frac{dT_v(\tau_0)}{dT_s} + \int_{T_v(\tau_0)}^{1} \pi \frac{dB_v(T)}{dT_s} dT_v,
\]

\[
= \pi \frac{dB_v(T_s)}{dT_s} T_v(\tau_0) + \int_{T_v(\tau_0)}^{1} \pi \frac{dB_v(T)}{dT_s} dT_v.
\]

Next, we make use of the fact that the atmosphere’s emission temperature is approximately constant with respect to the surface temperature (see Figure 3), \( dB_v(T)/dT_s \approx 0 \). In this case the integral in the equation vanishes and we are left with

\[
\frac{dF_+^\rho}{dT_s} \approx \pi \frac{dB_v(T_s)}{dT_s} T_v(\tau_0).
\]

We integrate this expression over all frequencies to arrive at the net feedback,

\[
\frac{d\text{OLR}}{dT_s} \approx \int_{0}^{\infty} \pi \frac{dB_v(T_s)}{dT_s} T_v(\tau_0) d\nu.
\]

Equation 7 describes the feedback of a moist atmospheric column. We relate this expression to the surface blackbody feedback \( d(\sigma T_s^4)/dT_s = 4\sigma T_s^3 \), which would be the column’s feedback if the atmosphere was transparent at all frequencies. We finally arrive at

\[
\frac{d\text{OLR}}{dT_s} \approx \int_{0}^{\infty} \pi \frac{dB_v(T_s)}{dT_s} T_v(\tau_0) d\nu.
\]

\[
\approx 4\sigma T_s^3 \times \int_{0}^{\infty} \frac{dB_v(T)}{dT_s} T_v(\tau_0) d\nu,
\]

\[
\approx 4\sigma T_s^3 \times \bar{T}.
\]

Here \( \bar{T} \) is the spectrally average of the transmission function between the surface and top-of-atmosphere, weighted by the temperature derivative of the Planck function, and is defined as

\[
\bar{T} = \int_{0}^{\infty} \frac{T_v}{dB_v(T_v)} d\nu.
\]

Equation 8 states that the total feedback of a moist atmospheric column is simply equal to the blackbody feedback of the surface, filtered through a partially-transparent atmosphere. If a frequency is optically thick its transmission vanishes, \( T_v \ll 1 \), so that only spectral window regions, where \( T_v \sim 1 \), contribute to the feedback. If the atmosphere is optically thin at all frequencies then \( \bar{T} = 1 \), and the feedback is simply equal to the surface feedback. Conversely, if the atmosphere becomes optically thick at all frequencies then \( \bar{T} \rightarrow 0 \), and the column’s feedback vanishes. This is the runaway greenhouse limit, in which the OLR asymptotes towards a constant value and becomes independent of surface temperature.

We note that Equation 8 does not state that atmospheric emission, and therefore the atmosphere’s radiative cooling, is constant with surface temperature. Instead the atmospheric emission increases with warming at a rate that is exactly cancelled by the decrease in surface emission. This can be seen from the line preceding Equation 5, which we reproduce here:

\[
\frac{dF_+^\rho}{dT_s} = \cdots + \pi B_v(T_s) \frac{dT_v(\tau_0)}{dT_s} - \pi B_v(T(\tau_0)) \frac{dT_v(\tau_0)}{dT_s} + \cdots
\]

As long as the difference between \( T_v \) and \( T(\tau_0) \) is small these two terms cancel and do not affect the net feedback. However, one needs to consider the terms separately if one wanted to consider the change in atmospheric emission versus the change in surface emission. In particular, even if the atmosphere’s emission temperature is fixed at a given frequency (see Fig. 3), surface warming still increases the atmosphere’s net radiative cooling. This is because surface warming allows the atmosphere to radiate from a wider range of frequencies due to the term \(-dT_v(\tau_0)/(dT_s) \times \pi B_v(T(\tau_0))\). Here \( dT_v(\tau_0)/(dT_s) \) is negative (transmission decreases with warming) so the overall term is positive.
We similarly evaluate WV as warming while keeping the relative humidity fixed. We note that our radiation code is written in pressure coordinates, which means a perturbation in temperature at a given optical thickness in turn can be written as a function of temperature and specific humidity, $T(\tau_s) = f(T, q)$. Intuitively, the dependence on $q$ arises because, even if temperature is held fixed in pressure coordinates, changes in humidity can still affect the value of $T$ at a given $\tau_s$. Next, we decompose the change in $T$ into a contribution due to the temperature effect of surface warming, and the humidity effect of surface warming. With some manipulation we arrive at

$$\frac{dT}{d\tau_s} \bigg|_{\tau_s} \approx \frac{T(T_s + \delta T_s, q) - T(T_s, q)}{\delta T_s},$$

where $\delta T_s$ denotes the surface component of the Planck feedback, $PL_{atm}$ the atmospheric component of the Planck feedback, LR the lapse rate feedback, and WV the water vapor feedback.

The four terms in Equation 12 correspond to the spectral feedbacks due to surface warming, uniform atmospheric warming, non-uniform atmospheric warming at fixed humidity (i.e., due to changes in the lapse rate), and changes in humidity. We integrate these terms over frequency,

$$PL_{surf} = \int_0^{\infty} d\nu \frac{dB_v(T_s)}{dT_s} e^{-\tau_0} \left( T \right),$$

$$PL_{atm} = \int_0^{\infty} d\nu \int_{\tau_0}^{\infty} d\tau_v \frac{dB_v}{dT_s} \left( T \right) e^{-\tau_v},$$

$$LR = \int_0^{\infty} d\nu \int_{\tau_0}^{\infty} d\tau_v \frac{dB_v}{dT_s} \left( T \right) \left( \frac{\partial T}{\partial T_s} \right)_q \left| \frac{\partial T}{\partial T_s} \right| \left( T \right) e^{-\tau_v},$$

$$WV = \int_0^{\infty} d\nu \int_{\tau_0}^{\infty} d\tau_v \frac{dB_v}{dT_s} \left( T \right) \left( \frac{\partial T}{\partial T_s} \right)_q \left| \frac{\partial T}{\partial T_s} \right| \left( T \right) e^{-\tau_v}.$$

Following Soden et al. (2008), for $\delta T$ we use an increase of 1 K and for $\delta q$ we use a moistening that corresponds to a 1 K warming while keeping the relative humidity fixed. We note that our radiation code is written in pressure coordinates, which means a perturbation in $T$ or $q$ modifies the values of optical thickness. To evaluate the above derivatives at constant optical thickness we compute $\tau_v$ thrice (for a reference state, as well as for the perturbations in temperature and humidity). We then use linear interpolation to translate $T(T_s + \delta T_s, q)$ and $T(T_s, q + \delta q)$ from the optical thickness of a perturbed state back to the optical thickness of the reference state. For numerical stability we interpolate on $\log(\tau_v)$ instead of $\tau_v$. 

4. Comparison with standard feedback decompositions.

It is common to decompose the net clear-sky feedback into Planck, water vapor, and lapse rate feedbacks (9, 10). Here we show that the sum of the atmospheric Planck feedback, water vapor feedback, and lapse rate feedback largely cancel to satisfy the constraint that, outside of window regions, the atmospheric longwave emission has to be roughly invariant with surface warming (Figure 3). Although Ingram performed a similar analysis (11), in this section we explicitly derive the feedback decomposition to clarify the underlying assumptions and implications. We begin with the contribution to the net feedback at a single frequency, Equation 5, which we write in optical thickness coordinates:

$$\frac{dF^+}{dT_s} = \pi \frac{dB_v(T_s)}{dT_s} e^{-\tau_0} + \int_0^{\tau_0} \pi \frac{dB_v(T(\tau_s))}{dT_s} e^{-\tau_v} d\tau_v,$$

The change in atmospheric emission with surface warming, $dB_v(T)/dT_s$, is solely a function of atmospheric temperature $T$. The four terms in Equation 12 correspond to the spectral feedbacks due to surface warming, uniform atmospheric warming, non-uniform atmospheric warming at fixed humidity (i.e., due to changes in the lapse rate), and changes in humidity. We use linear interpolation to translate $\delta T$ and $\delta q$ from the optical thickness of a perturbed state back to the optical thickness of the reference state. For numerical stability we interpolate on $\log(\tau_v)$ instead of $\tau_v$. 

Daniel D.B. Koll, Timothy W. Cronin 5 of 15
The circles in Figure 3 show the four feedback components in a H$_2$O-saturated atmosphere at a surface temperature of 285 K. In accordance with the constraint that the atmospheric emission temperature has to be invariant with surface warming, PL$_{atm}$, LR, and WV largely cancel. This means the net feedback is dominated by the surface component of the Planck feedback, PL$_{surf}$, which is captured by our simple feedback model.

Similarly, Figure S6 shows the feedback decomposition for an atmosphere with 400 ppm of CO$_2$ and varying amounts of relative humidity. At surface temperatures less than ~300 K the net feedback is again dominated by PL$_{surf}$, although the cancellation between PL$_{atm}$, LR, and WV becomes less effective at temperatures exceeding ~300 K. Nevertheless, the maximum value of PL$_{atm}$+LR+WV never exceeds the maximum value of PL$_{surf}$. This implies that the overall temperature-dependence of the net feedback, and thus also the near-linearity of OLR, is dominated by the temperature-dependence of PL$_{surf}$.

5. Shape of the transmission function.

To understand the shape of transmission in Figure 4, we first consider an atmosphere in which the window region is close to the spectral peak of the Planck derivative, $dB_v/dT_s$, such that $dB_v/dT_s$ can be approximated as constant with frequency $\nu$. This is approximately the case for H$_2$O (see Figure 3). Alternatively, one can consider an atmosphere with a window region that is narrow compared to the spectral width of $dB_v/dT_s$. In both cases $dB_v/dT_s$ can be removed from the integral which defines the transmission $\mathcal{T}$, so Equation 9 becomes

$$\mathcal{T} = \frac{\int_0^\infty T_v \frac{dB_v(T_v)}{dT_s} d\nu}{\int_0^\infty \frac{dB_v(T_v)}{dT_s} d\nu}$$

$$= \frac{1}{4\pi T_s^2} \times \pi \int_0^\infty T_v \frac{dB_v(T_v)}{dT_s} d\nu,$$

$$\approx \frac{1}{4\pi T_s^2} \times \left( \frac{dB_v}{dT_s} \right) \int_0^\infty T_v(T_0) d\nu,

$$\approx \frac{1}{4\pi T_s^2} \times \left( \frac{dB_v}{dT_s} \right) \int_{\nu_0}^\infty T_v(T_0) d\nu,$$

where $\nu_0$ describes the center of the window region, and $\Delta \nu(T_s)$ is the spectral width of the window region. The temperature dependence of the normalized Planck derivative, $(4\pi T_s^2)^{-1} \times \pi dB_v/dT_s$, is small relative to that of $\Delta \nu(T_s)$ which can be understood by considering how the amplitude and the shape of $\pi dB_v/dT_s$ change with temperature. The amplitude of the Planck derivative $\pi dB_v/dT_s$ increases rapidly with temperature, but this change is largely captured by the normalization $4\pi T_s^2$. Similarly, the spectral shape of $\pi dB_v/dT_s$ is a relatively weak function of temperature; for example, Wien’s displacement law entails that the peak frequency of $\pi dB_v/dT_s$ only shifts linearly with temperature. In contrast, $\Delta \nu(T_s)$ varies by order unity across the range of temperatures we consider. Therefore $\mathcal{T}$ depends on temperature primarily through $\Delta \nu(T_s)$.

To understand how $\Delta \nu(T_s)$ changes with temperature, we consider a band model in which the absorption cross-section decays exponentially away from a band center. Supplementary Figure S7 shows that this assumption is valid for a number of molecules, including H$_2$O, CO$_2$, and NH$_3$. For such a band $\kappa_0 = \kappa_0 e^{-A(\nu-\nu_0)}$, where $\nu_0$ denotes the center of the band and $\kappa_0$ and $A$ are constants. The atmosphere’s water vapor path increases roughly exponentially with temperature, which we express as $WVP(T_s) = WVP(T_0) e^{B(T_s-T_0)}$, where $WVP(T_0)$, $B$, and $T_0$ are constants. The window region is defined by those frequencies in which the atmosphere stops being optically thick, with the edge of that region defined as $\tau \sim 1$. We denote the frequency at this edge as $\tilde{\nu}$ so that

$$\tau \sim \kappa_0 \times WVP,$$

$$1 \sim \kappa_0 \times WVP,$$

$$1 \sim \kappa_0 e^{-A(\tilde{\nu}-\nu_0)} \times WVP(T_0) e^{B(T_s-T_0)}.$$

To satisfy this condition the exponents must sum to a constant, which means the edge of the window region has to be a linear function of temperature,

$$A(\tilde{\nu} - \nu_0) = B(T_s - T_0) + \text{const},$$

$$\tilde{\nu}(T_s) = \frac{B}{A} (T_s - T_0) + \text{const}.$$

Now, if the edges of the window region are linear functions of temperature, then its total width also has to be linear, $\Delta \nu(T_s) = \Delta \nu(T_0) - \tilde{\nu}(T_s)$. It follows that the transmission should decrease linearly with temperature,

$$\mathcal{T} \propto \Delta \nu(T_s),$$

$$\mathcal{T} \propto \text{const} - \frac{B}{A} (T_s - T_0).$$

Supplementary Figure S7 (bottom row) supports this simple model, and shows that the transmission for several species decreases approximately linearly with temperature. A linear decrease in transmission holds particularly well at low temperatures, when
continuum absorption can be neglected. For example, we computed the full transmission of H$_2$O as well as the transmission of H$_2$O that is solely due to line absorption (full versus dashed line in Figure S7, bottom left panel). Without continuum absorption the transmission decreases linearly between 200 K and 350 K. In contrast, the full transmission starts deviating from a linear slope due to H$_2$O continuum absorption above $\sim$ 270 K, and then bottoms out above $\sim$ 300 K.

6. Climate feedback for a linear transmission function.

If the transmission decreases exactly linearly between $T_0$ and $T_\infty$, then the feedback is equal to

$$\lambda = \frac{d\text{OLR}}{dT_s} = 4\sigma T_s^4 \times \frac{T_\infty - T_s}{T_\infty - T_0}.$$  \[18\]

By taking the derivative of this equation with respect to temperature, we find that the maximum of $\lambda$ occurs at a temperature $3/4 \times T_\infty$, so long as $T_0 < 3/4 \times T_\infty$ (this assumption is valid for H$_2$O; see Fig. S7). To find the temperature range over which $\lambda$ is constant to within $\pm$10%, we solve for the range over which $\lambda$ is equal to within $\epsilon = 0.9 \times 0.9 = 81.8\%$ of its maximum value at $3/4 \times T_\infty$. This temperature range is determined by $\lambda(T)/\lambda(3/4 \times T_\infty) \geq \epsilon$ which results in

$$\frac{4^4}{3^3} \left( \left( \frac{T}{T_\infty} \right)^3 - \left( \frac{T}{T_\infty} \right)^4 \right) \geq \epsilon.$$  \[19\]

We solve this equation numerically, and find that the feedback is approximately constant over the temperature range $0.6 \leq T/T_\infty \leq 0.87$. Over this range, the feedback is approximately equal to

$$\lambda \approx 0.9 \times \lambda_{\text{max}} = 0.9 \times \left( \frac{3}{4} \right)^3 \sigma T_\infty^3 \frac{T_\infty}{T_\infty - T_0}.$$  \[20\]

For H$_2$O without continuum absorption $T_0 \sim 190K$ and $T_\infty \sim 350K$ (Fig. S7), so the feedback would be approximately constant between 210 K and 305 K and equal to $\lambda \approx 2$ W m$^{-2}$ K$^{-1}$. In reality the range over which Earth’s feedback is almost constant is smaller, and the feedback slightly larger, because transmission is not exactly linear with temperature and because continuum absorption acts to rapidly reduce the feedback at high temperatures (Figs. 2,5).
Fig. S1. Sensitivity of OLR to different assumptions about the average zenith angle used in the two-stream equations. The left panel shows the OLR for an Earthlike atmosphere (1 bar dry gas, H$_2$O at saturation) computed using a number of different zenith angles. The right panel shows the differences in OLR relative to the default value used in our calculations. For comparison, red crosses show results using DISORT, a multi-stream discrete ordinate solver.
Fig. S2. Validation of our code. Left column shows OLR computed with our code and using the same temperature-pressure profiles as in Goldblatt et al, compared to the results from that paper. Right column shows the differences in OLR. Top row shows the steam atmosphere limit (pure H$_2$O), bottom row shows the Earthlike limit (H$_2$O plus 1 bar of N$_2$). The largest differences amount to $\sim 1.4$ W m$^{-2}$ in the steam limit, and $\sim 6$ W m$^{-2}$ in the Earthlike limit at surface temperatures of $\sim 350$ K.
Fig. S3. Sensitivity of OLR to different assumptions about the H$_2$O continuum used in our calculations. Top row: Absorption cross-sections from the MTCKD model, the Baranov et al laboratory measurements, and our hybrid model (see Supplementary Text). Shown are outputs for two representative values of pressure and temperature. Bottom left: OLR computed with our code using the MTCKD model (blue) and our hybrid model (red). Bottom right: The maximum difference in OLR between the two calculations amounts to about 1 W m$^{-2}$. 
Fig. S4. The thermodynamic and radiative effect of H$_2$O is distinct from that of CO$_2$. A dry atmosphere with 400ppm of CO$_2$ exhibits a clearly non-linear OLR (red), similar to that of a blackbody (black). In contrast, an atmosphere with fixed relative humidity (blue) develops an OLR that is approximately linear such that its feedback only changes by ±10% over a wide range of temperatures (thick line).
Fig. S5. OLR and feedback for an atmosphere with 400 ppm of CO₂ and varying relative humidity (RH). The dashed lines are a linear fit (left) which implies a constant feedback (right), and show the range over which the feedback changes by less than ±10%. These ranges span 65 K at RH=100%, 70 K at RH=50%, 65 K at RH=30%, and 30 K at RH=10%. The corresponding feedbacks are, to within ±10%, 1.9 W m⁻² K at RH=100%, 2.2 W m⁻² K⁻¹ at RH=50%, 2.5 W m⁻² K⁻¹ at RH=30%, and 3.6 W m⁻² K⁻¹ at RH=10%.
Fig. S6. Feedback decomposition for an atmosphere with 400 ppm of CO$_2$ and varying relative humidity. Solid lines are the net feedback, symbols show the contributions due to the atmospheric Planck, surface Planck, lapse rate, and water vapor feedbacks (see Supplementary Text). Up to surface temperatures of ≲ 300 K the net feedback is dominated by the surface contribution to the Planck feedback, whereas the three other feedbacks largely cancel; the cancellation arises because the atmosphere’s emission temperature is roughly invariant with surface warming (Fig. 3). In addition, the temperature-dependence of the net feedback, including its near-constancy over a wide range of temperatures, is dominated by the temperature-dependence of the surface term.
Fig. S7. Top row: H$_2$O, CO$_2$, and NH$_3$ all have absorption bands that decrease roughly exponentially away from band centers. Cross-sections are evaluated at temperatures indicated by orange stars (bottom row) and are smoothed using a median filter of width 10 cm$^{-1}$. Bottom row: The exponential shape of absorption bands makes transmission decrease approximately linearly with temperature (see Supplementary Text). A linear decrease is particularly clear for H$_2$O without continuum absorption (dashed line) and NH$_3$. For CO$_2$, line absorption also leads to a quasi-linear decrease (dashed line) but the lines do not cover the entire spectrum and thus saturate at $\bar{T} \sim 0.35$. At high temperatures continuum absorption becomes additionally important, and transmission decreases rapidly. This occurs for H$_2$O above 280 K, and for CO$_2$ above 170 K.
References