GeoSci 236: Demonstration of Some Eigen-Techniques

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Recall that with the 2 data matrices

$$\mathbf{A}_{M\times N} \equiv \begin{bmatrix} \vdots & \vdots & & \vdots \\ \mathbf{a}(t_1) & \mathbf{a}(t_2) & \cdots & \mathbf{a}(t_N) \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

and

$$\mathbf{B}_{P\times N} \equiv \begin{bmatrix} \vdots & \vdots & & \vdots \\ \mathbf{b}(t_1) & \mathbf{b}(t_2) & \cdots & \mathbf{b}(t_N) \\ \vdots & \vdots & & \vdots \end{bmatrix},$$

the cross-covariance matrix is

$$\mathbf{C} = \sum_{i=1}^{N} \begin{bmatrix} \vdots \\ \mathbf{a}_i \\ \vdots \end{bmatrix} \begin{bmatrix} \cdots & \mathbf{b}_i^T & \cdots \end{bmatrix} = \begin{bmatrix} \sum a_i(1)b_i(1) & \sum a_i(1)b_i(2) & \cdots & \sum a_i(1)b_i(P) \\ \sum a_i(2)b_i(1) & \sum a_i(2)b_i(2) & \cdots & \sum a_i(2)b_i(P) \\ & & \vdots & \\ \sum a_i(M)b_i(1) & \sum a_i(M)b_i(2) & \cdots & \sum a_i(M)b_i(P) \end{bmatrix} =$$

$$= \begin{bmatrix} \sum b_i(1) \begin{pmatrix} \vdots \\ \mathbf{a_i} \\ \vdots \end{pmatrix} & \sum b_i(2) \begin{pmatrix} \vdots \\ \mathbf{a_i} \\ \vdots \end{pmatrix} & \cdots & \sum b_i(P) \begin{pmatrix} \vdots \\ \mathbf{a_i} \\ \vdots \end{pmatrix} \end{bmatrix},$$

where all sums span 1 through N, N being the time dimension **A** and **B** share. The purpose of this section of the notes is to give you a feel for how the covariance behaves under various conditions.

1 Example 1

All is very simple when both **A** and **B** are nothing but noise, both in time and in space, pretty much regardless of whether the noise is white of Gaussian. The covariance matrix shows no structure whatsoever, and the falloff of the singular spectra is very slow (there is very little difference between lowest and highest singular values).

Figure 1: Two covariance matrices of 2 noise datasets. The left panel shows covariance of white noise, while the right is for Gaussian noise. The Singular spectra of both are shown in the lower panel.

Figure 2: Patterns used to generate the datasets of Example 2.

Figure 3: Covariance matrices of 4 realizations of the random process Example 2 is based on.

2 Example 2

Things begin to get hairier when the datasets consist of structured spatial patterns. Let the spatial patterns be given by

$$\mathbf{P}_1 = cos\left[\frac{2\pi x_1}{30}\right] cos\left[\frac{2\pi y_1}{14}\right], \qquad \mathbf{P}_2 = 2\ exp\left[\frac{-x_1^2}{40}\right] exp\left[\frac{-(y_1-7)^2}{14}\right],$$

$$\mathbf{S}_1 = cos\left[\frac{2\pi x_2}{11}\right] cos\left[\frac{2\pi y_2}{5}\right], \qquad \mathbf{S}_2 = 2\ exp\left[\frac{-(x_2-11)^2}{20}\right] exp\left[\frac{-y_2^2}{7}\right],$$

shown in Fig. 2 for $x_1 = 0 \cdots 15$, $y_1 = 0 \cdots 7$, $x_2 = 0 \cdots 11$ and $y_2 = 0 \cdots 5$. The temporal behavior of both datasets is a Gaussian random combination of the 2 patterns. [That is,

$$\mathbf{A} = \alpha(t)\mathbf{P}_1 + \beta(t)\mathbf{P}_2,$$

$$\mathbf{B} = \gamma(t)\mathbf{S}_1 + \xi(t)\mathbf{S}_2,$$

with $\alpha \sim \beta \sim \gamma \sim \xi \sim N(0,1)$ in time.]

Given the stochastic nature of the process, Fig. 3 shows 4 different realizations of the process. While the realizations differ from one another, it is clear that some regions of **C** tend to have larger cross-covariances than others. The regions of consistently large cross-covariances correspond to products of grid points of **A** and **B** that are most likely to have large, like-sign, entries. For example, some elements in the lower-left corners of all 4 **C**s are rather high, corresponding to positive superpositions of high values found in the corners of both domains. Let's look at the

Figure 4: Rows 7-10 of the cross-covariance matrix of realization 4 in Example 2.

highest cross-covariance of the 4th realization, $\mathbf{C}_{8,69}$ (near $\mathbf{C}$'s lower-right corner). This element corresponds to the cross-covariance of $\mathbf{A}$'s 8th grid point and $\mathbf{B}$'s 69th. Given the geometry of the domains, these are $\mathbf{A}$'s upper-left corner (exactly) and $\mathbf{B}$'s lower-right corner (approximately; it is in fact 3 grid points above the corner, representing a slight bias toward a negative representation of one of the basis patterns during the construction of $\mathbf{B}$). To further clarify the point, Fig. 4 shows 4 lines along $\mathbf{C}$'s dimension which corresponds to the space dimension of $\mathbf{B}$. Notice the sign-shift between lines 8 and 9. This corresponds to element 8 representing $\mathbf{A}$'s upper-right corner, while element 9 is represents $\mathbf{A}$'s next element, which is along the *lower* boundary.

Next, Fig. 5 shown the 4 singular spectra. Note that despite using 2 equal-amplitude patterns in constructing both $\mathbf{A}$'s and $\mathbf{B}$, $\mathbf{C}$ is very strongly dominated by a single mode. Also, note there are 3 (rather than 2) non-zero modes. Both are, to some degree, manifestations of random alignment of noise. The spatial patterns can be revealing. Note the way the leading mode is sometime dominated by one basis pattern, some other time by the other, and, in general, how the 2 blend together to yield the leading mode. The same also holds for $\mathbf{B}$. This can perhaps become clearer when viewing the sum and difference of the pairs, shown in Fig. 8

The 2nd modes, the only modes other than the leading ones that still possibly hold useful information, are shown in Figs. 9 and 10. They too clearly show the basis patterns, along with some alignments of crests and troughs having to do with random fluctuation in the time behaviors.

Figure 5: Singular spectra of the 4 realizations in Example 2.

Figure 6: Spatial structure of the leading mode, **A**.

Figure 7: Spatial structure of the leading mode, **B**.

3 Example 3

Now let's consider **A** and **B** arising from the same spatial patterns as before (Fig. 2), but with structured time evolution. That is, with

$$\mathbf{P}_1 = cos\left[\frac{2\pi x_1}{30}\right] cos\left[\frac{2\pi y_1}{14}\right], \qquad \mathbf{P}_2 = 2\ exp\left[\frac{-x_1^2}{40}\right] exp\left[\frac{-(y_1-7)^2}{14}\right],$$

$$\mathbf{S}_1 = cos\left[\frac{2\pi x_2}{11}\right] cos\left[\frac{2\pi y_2}{5}\right], \qquad \mathbf{S}_2 = 2\ exp\left[\frac{-(x_2-11)^2}{20}\right] exp\left[\frac{-y_2^2}{7}\right]$$

as before (Fig. 2), and

$$x_i = \begin{cases} 0\cdots 15 & i=1 \\ 0\cdots 11 & i=2 \end{cases} \qquad y_i = \begin{cases} 0\cdots 7 & i=1 \\ 0\cdots 5 & i=2 \end{cases},$$

we now let the fields evolve according

$$\mathbf{A} = cos\left(\frac{2\pi t_i}{30}\right)\mathbf{P}_1 + sin\left(\frac{2\pi t_i}{30}\right)\mathbf{P}_2 \qquad t = 1\cdots 30$$

$$\mathbf{B} = cos\left(\frac{2\pi t_i}{30}\right)\mathbf{S}_1 + sin\left(\frac{2\pi t_i}{30}\right)\mathbf{S}_2 \qquad t = 1\cdots 30.$$

The results are shown in Figs. 11 and 12. Now the dominance of the cross-covariance of **A**'s upper-left corner and **B**'s lower-right corner is abundantly clear, as reflected in both the plot of

Figure 8: The sums (lower panels) and differences (upper panels) of the 2 patterns used to generate **A** (left panels) and **B** (right panels).

Figure 9: Spatial structure of the 2nd mode, **A**.

Figure 10: Spatial structure of the 2nd mode, **B**.

Figure 11: Results of Example 3. The upper panels show the cross-covariance matrix and its singular spectrum, while the lower panels offer cross-sections along **C** at and near the highest absolute cross-covariance.

Figure 12: The 1st and 2nd modes corresponding to both datasets in Example 3.

C itself, as well as of the 2 cross-sections along it. This is also reflected in the mode structures. The leading mode of **A** is clearly completely dominated by the upper-left corner, where the 2 basis patterns have a positive superposition more often than not. Adding to this is the fact that **A**'s 2nd pattern, $\mathbf{P}_2$, has a Frobenius norm almost two-thirds larger than that of $\mathbf{S}_1$. The 2nd mode is mostly concentrated at the 3 remaining corners, where one basis pattern has high loadings, with very little chance for the 2nd pattern to have a negative superposition with it.

The leading mode of **B** is primarily focused at the lower-right corner. This can be compared with Fig. 8, where it is seen that both $\mathbf{S}_1 \pm \mathbf{S}_2$ have a maximum amplitude there. The remainder of the modes' structures appear to be principally influenced by $\mathbf{S}_1$. This, again, is partly due to the smaller Frobenius norm of $\mathbf{S}_1$ as compared with $\mathbf{S}_2$'s.