Bernoulli Equation for Energy in a Fluid:

\[ J/\text{kg} = \frac{1}{2} v^2 + g h + P/\rho = \text{constant} \]

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top mass \( \text{E.den} = gh \) ❯ Dam

WATER

bottom mass

\( \text{E.den} = P/\rho \) ❯ dam outflow mass \( \text{E.den} = \frac{1}{2} v^2 \)

Proof that Energy Density is Equal for all Dammed Water:

Pressure is Force/Area and Force is mass * acceleration

For our kg of water at the bottom of the dam, the mass of water above it is equivalent to the volume of water above * density

\[ m = V \times \rho \]

And the Volume is the Area * height, making \( F = V \times \rho \times g \)

Therefore, Pressure = \( A \times h \times \rho \times g / A \)

Now if we cancel A, \( P = \rho \times g \times h \).

In our energy density equation, the pressure energy density is \( P/\rho \).

Therefore, \( P/\rho = g \times h \)

No matter what height the water is at, it still has the same energy density.

Why build Dams?:

In free flowing rivers, you are flowing downhill and therefore losing potential energy.

However, this potential energy is not converted to kinetic energy because the velocity of your river on top of the incline is usually equal to or greater than the velocity at the bottom of the incline (i.e. Kinetic energy stays constant).

Instead, this energy is lost from the system by frictional forces and turbulence.

This is a violation of Bernoulli’s equation because energy is leaving the system.

Dams are able to harness this lost energy by eliminating friction from the flowing of the river.

Energy density of Hoover Dam water: 2000 J/kg
Energy density of gasoline: 44 MJ/kg
Energy density of free flow river (v = 3m/s): 5 J/kg

Dams therefore are able to use the 2000 J/kg of river potential energy that is lost when the river flows. “Run of River” throws this away.

**Relating Energy Density to Power:**

Energy density = Energy/mass

Power = Energy / time = (Energy / mass) * (mass / time)

To convert between energy density and power we need to multiply by a factor of mass/time.

This is known as the mass flow rate, or $Q_m$.

A note about $Q_m$: 
To a thermal engineer, Q is heat flow (in J/s)

To a hydro engineer, Q_m is mass flow (in kg/s)

What is Q_m?

To find this we might need to know the density, volume and velocity.

We know that Q_m = \rho * Q_v where Q_v is the volume flow rate in m^3/s.

To get m^3/s, you can take a velocity and multiply it by an area (Q_v = v * A)

Putting the two equations together yields Q_m = \rho * v * A

Therefore, Power = Energy density * Q_m = E. den. * \rho * v * A

“Run of River” and Free Stream:

“Run of River” means there is no modification to the entire stream but there can be modification to the flow into the turbine (i.e. a Pelton Wheel in a river)

Free Stream means there is no modification to the river or the flow into the turbine (i.e. Wind)

The power you can get out of a free stream turbine:

Power = Q_m * Energy density = m/time * E/m

Q_m = \rho * A * v

E. den. = \frac{1}{2} v^2

Thus, Power = \frac{1}{2} * \rho * A * v^3

There is a huge dependence on the velocity of the flow

Why are Wind Turbines so Big?: 
• For the same wind speed, the bigger the area the more power they generate
• They need to be big to produce reasonable power with the low density of air
• There is more wind at higher altitude (wind speed has an exponential dependence on height) and therefore the turbines must be tall
• If you are going to be tall you should also be big around to get the most out of your materials

**Efficiency of windmills and Betz’s Law:**

Betz’s Law states the efficiency of extracting mechanical work when conditions are optimal

\[
\text{Physical Limit} = \text{Betz’s Law}\]

Betz’s Law says you can only be 0.59 efficient

This is the limit for free stream extraction

You can’t be 100% efficient because you would stop the wind flow. This would generate the maximum power calculated above.

The windmill instead alters the streamlines around it so that the area of a flow is larger upon exiting the turbine than entering. This change in area slows down the wind.

The effective limit occurs when \( \frac{v_{\text{out}}}{v_{\text{in}}} = 1/3 \).
Don’t be fooled by assuming that if $v_{out}/v_{in} = 1/3$ then the power must be somehow reduced by $(1/3)^3$. The velocity that the turbine sees is neither $v_{in}$, the wind speed well upstream of the turbine, nor $v_{out}$, the wind speed well downstream of the turbine. Instead it is exactly in between (here marked $v_{avg}$).

That means that the turbine sees velocity that is $(1 + 1/3)/2$ of the environmental velocity, or $(4/3)/2 = 2/3$ of the average.

The extractable power can be shown to be equal to the free-stream power times a factor of

$$4 a (1-a)^2$$

where $a = v_{out}/v_{in}$. That factor is then $(4/3)*(2/3)^2 = (4/3)*(4/9) = 16/27 \sim 0.59$

This is Betz’s Law, which says that we can only extract 59% of the power we calculated above for free stream turbines. (And this is only for perfect turbines with constant wind). (For a full proof of Betz’s law you need to do a longer control volume analysis; see the reading on turbine aerodynamics).

In real-world use, windmills are typically 15% efficient, and therefore produce only 15% of the average max power because of variable wind speed conditions. They can approach Betz’s law when the wind is right, but are less efficient and even shut down if wind is too high or too low.