

## The Bernoulli equation and the energy content of fluids

What turbines do is to extract energy from a fluid and turn it into rotational kinetic energy, i.e. spin something, almost always for the purposes of generating electricity. Turbine shape and design are governed by the characteristics of the fluid: its energy content, density, and flow rate. Hydro turbines are the most diverse in design because hydropower can be extracted in a wide range of conditions. Roughly speaking, the higher the energy content of the fluid, the more the hydro turbines used start to look like gas turbines, and the lower the energy content of the fluid, the more the hydro turbines start to look like windmills.

The Bernoulli equation describes how much energy a fluid carries, that can potentially be extracted as mechanical work:

$$E_{\text{total}} = KE + PE + PR + U + L$$

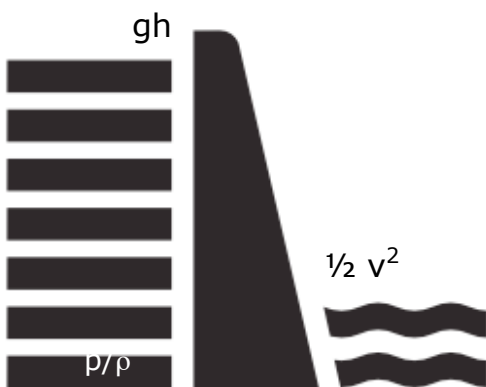
where KE is kinetic energy ( $\frac{1}{2} v^2$ ), PE is potential energy ( $g \cdot h$ ), PR is “pressure energy” ( $p/\rho$ , where  $\rho$  is density) U is internal energy ( $c_v T$ , where  $c_v$  is the specific heat at constant volume), and L is the latent heat of vaporization (ignored unless we’re dealing with steam turbines; no others involve phase changes). (Here the equation is written in energy per unit mass units, or J/kg).

Why is Bernoulli’s equation useful? Because conservation of energy means that (at least until the fluid starts doing work in your turbine), the total energy is constant no matter what you do to any one variable. Energy can be swapped back and forth between the terms.

**For an incompressible fluid, like water,** regardless of what one does, the internal energy remains constant, i.e. temperature does not change. Water is not perfectly incompressible, but it is nearly so, enough that we can neglect the internal energy term for any turbine problem. That means that if there is no heat added or removed from the system, and no work done on or by it, conservation of energy reduces to:

$$\frac{1}{2} v^2 + g \cdot h + p/\rho = \text{ct.}$$

Incompressibility also means that density  $\rho$  is constant. (In fact, that’s the intuitive idea of what it means to be not compressible.) Bernoulli’s equation then represents a tradeoff between  $v$ ,  $h$ , and  $p$ . (For water, remember  $\rho = 1 \text{ g/cm}^3$ ).



As an illustration, think of a parcel of water behind a dam. We can readily move that parcel up or down in the water behind the dam, or we can poke a hole in the dam and let it flow out – all without adding energy or doing work. At the top of the dam, the parcel has no velocity and its pressure energy is effectively zero, but it has maximum potential energy ( $gh$ ). At the bottom of the dam, it has no velocity and no potential energy but maximum pressure energy ( $p/\rho$ ). When squirting out it has effectively no pressure energy and no potential energy but

maximum kinetic energy ( $\frac{1}{2} v^2$ ). But in no case did the parcel change its total energy - it simply swapped energy from one term to the other. Because energy is conserved, if you know the head ( $h$ ) of the dam, you know the energy content of water everywhere in the dam system. By the Bernoulli equation, you also know the pressure at the bottom, or the maximum velocity of the water exiting the dam.

**Turbine classes:** Although turbines are very diverse, they can broadly be sorted into two major classes, “impulse” and “reaction” turbines.

- An impulse turbine extracts energy by reducing the fluid velocity ( $\frac{1}{2} v^2$  term)
- A reaction turbine extracts energy by reducing the fluid pressure ( $p/\rho$  term)

The *Francis turbine* is an example of a pure reaction turbine. Pressure is dropping through the Francis turbine but velocity is constant (and slow) in the Francis turbine, until the water hits the draft tube (also called the “tail race”). (The draft tube is designed to wring that last bit of energy out of the fluid, slowing the water down and creating even lower pressure to get one additional “pull” of water through the turbine).

The *Pelton wheel* is an example of a pure impulse turbine. All the water’s energy is converted to kinetic energy by passing it through a nozzle to make a high-speed jet of water that impacts the wheel and exerts force on it. The wheel turns in air and the whole system is at atmospheric pressure; no pressure gradients are involved.

**For a compressible fluid, like air**, the Bernoulli equation still holds, but density  $\rho$  is not constant. That means the internal energy term can’t be neglected, because you can swap energy between internal heat and pressure. Applying the ideal gas law in the form  $P = \rho \cdot RT$  and using the definitions  $\gamma = c_p/c_v$  and  $c_p = c_v + R$ , the Bernoulli equation reduces readily to:

$$\frac{1}{2} v^2 + g \cdot h + \frac{p}{\rho} \cdot \frac{\gamma}{(\gamma-1)} = \text{ct.}$$

where  $\gamma$  is a characteristic of the fluid. (See the solutions of the Carnot problem set for more explanation). The term  $\gamma$  for air is  $\sim 1.4$ . If you did engineering on a natural gas turbine, this is the equation you would have to use.

**For a compressible fluid that also changes phase, like saturated steam** we’d have to include a latent heat term too in the original conservation of energy equation.

### **Effective head**

The Bernoulli equation as written above is in terms of mass energy density: each term has the units J/kg. But these may not be the right units for the purposes of thinking about practical energy generation, because you often care more about volume than about mass. Consider: wind flowing at 10 m/s has the exact same mass energy density J/kg as water flowing at 10 m/s, but from a turbine of a given physical size you can extract a lot more from a flow of water than from one of wind. Why? Because the water has much more mass for a given volume.

In many situations, you don't care how much your working fluid weighs, but you care a lot about how much space it takes up. Engineers therefore often prefer to think not in terms of mass energy density (J/kg) but in terms of volume energy density (J/m<sup>3</sup>). On PS 1 you calculated the mass and volume energy density for many different potential combustible fluids and for batteries. In those cases the distinction didn't matter much for your ranking of a fuel, because the densities of all the various fuels and energy storage materials were roughly similar: gasoline was a bit less dense than water, and batteries a bit more dense, but the difference wasn't extreme. When you compare water and wind, though, the difference *is* extreme. Air near the surface is 1000 times less dense than water:  $\rho_{\text{air}}$  is about .001 g/cm<sup>3</sup>. A flow of air at normal atmospheric pressure therefore carries a thousand times less power than a flow of water through a turbine of equal size.

To convert the Bernoulli equation from J/kg to volume energy density (J/m<sup>3</sup>) we have to multiply through by something that has units kg/m<sup>3</sup>: the density,  $\rho$ .

$$\frac{1}{2} \rho \cdot v^2 + \rho \cdot g \cdot h + p = \text{ct.}$$

But, energy densities in J/m<sup>3</sup> are not really intuitive. For this reason engineers sometimes convert further to "effective head": that is, they determine the height of a dam that would produce water with the same volume energy density as whatever the working fluid is. (Dam height is termed "head"; conveniently  $h$  works as a symbol for either "height" or "head"). That's very intuitive over a pretty big range. People can easily visualize a dam of any height from centimeters to a kilometer.

Usually "effective head" is used to relate the kinetic energy in a flow of water or air to the energy content of water stored behind a dam. That kinetic energy is

$$\text{KE} = \frac{1}{2} \rho_{\text{fluid}} \cdot v^2$$

To find the effective head  $h_{\text{eff}}$ , equate this KE with the potential energy of water behind a dam, which is

$$\text{PE} = \rho_{\text{water}} \cdot g \cdot h_{\text{eff}}$$

So

$$\begin{aligned} h_{\text{eff}} &= \frac{1}{2} \rho_{\text{fluid}} \cdot v^2 / \rho_{\text{water}} \cdot g \\ &= \frac{1}{2} (v^2/g) \cdot (\rho_{\text{fluid}} / \rho_{\text{water}}) \end{aligned}$$

That permits easy comparison of energy extractable from a flowing fluid, regardless of its density. Since air is 1000 times less dense than water, the "effective head" of a stream of air will be 1000 times less than one of water moving at equal velocity.