On the thickness ratio in the quasigeostrophic two-layer model of baroclinic instability

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Abstract

It is shown that the classical quasigeostrophic two-layer model of baroclinic instability possesses an optimal ratio of layer thicknesses that maximizes the growth rate, given the basic-state shear (thermal wind), beta, and the mean Rossby radius. This ratio is interpreted as the vertical structure of the most unstable mode. For positive shear and beta, the optimal thickness of the lower layer approaches the mid-height of the model in the limit of strong shear/weak beta but it is proportional to shear divided by beta in the opposite limit. For a set of parameters typical of the Earth’s midlatitude, the growth rate maximizes at a lower-layer thickness substantially less than the mid-height and at a correspondingly larger zonal wavenumber.

It is demonstrated that a turbulent baroclinic jet whose statistical steady state is marginally critical when run with equal layer thickness can remain highly supercritical when run with a nearly optimal thickness ratio.
1. Introduction

The quasigeostrophic two-layer model, developed more than 60 years ago by Phillips (1951), contains all essential ingredients of large-scale dynamics in the midlatitude yet its solutions are accessible through elementary mathematics and affordable numerical integrations. Held (2005) refers to the model as the ‘E. Coli of climate models,’ drawing analogy to the guiding roles played by the microorganism in molecular biology.

In application to atmospheric baroclinic instability, the rest thicknesses of the two layers are often assumed equal. Since the boundary of the upper and lower troposphere is vague at best, this may be considered an unbiased choice. However, the thickness ratio affects the dynamics of baroclinic instability and therefore the consequence of its choice merits careful examination. Despite the thorough documentation of the unstable normal modes (Phillips 1954; Pedlosky 1987 §7.11; Vallis §6.6), we feel that this aspect of Phillips’s model has received little attention and hence the writing of this note.

2. Two-layer model with unequal layer thicknesses

The thickness ratio in Phillip’s model affects baroclinic instability in two ways. First, it sets the steering level of an unstable mode, where the phase speed of the mode equals the basic-state zonal wind. In the two-layer model all unstable modes are steered at the layer interface because, according to the semi-circle theorem, the phase speed of any unstable mode must lie between the maximum and minimum values of the basic-state zonal wind (Pedlosky 1987 §7). If we define the penetration depth of a mode using the steering level, the thickness of the lower layer provides exactly that depth.

Second, thickness ratio regulates the gradients of the basic-state potential vorticity
that the modes recognize. Suppose that beta and shear (the basic-state thermal wind) are both positive so their contributions to the lower-layer PV gradient are opposite in sign. Since modes recognize only the *vertically averaged* PV gradient in each layer, shear’s contribution is ‘diluted’ as the thickness of the layer increases. As a result, the critical shear necessary to keep the lower-layer PV gradient negative (and hence the flow unstable; Charney and Stern 1962) increases with the thickness of the lower layer.

Our main question is at what thickness ratio (vertical structure of the mode) the growth rate of instability is maximized given shear, beta, and other parameters of the flow. To address this question, we start from the standard, linear normal-mode PV equations and nondimensionalize them in a form convenient for our purpose:

\[
\begin{align*}
(U_1^* - c^*)\left[-(k^2 + l^2)\psi_1^* + \frac{f_0^2}{g''H_1^*}(\psi_2^* - \psi_1^*)\right] + \left(\beta^* + \frac{2f_0^2\Delta^*}{g''H_1^*}\right)\psi_1^* &= 0, \\
(U_2^* - c^*)\left[-(k^2 + l^2)\psi_2^* - \frac{f_0^2}{g''H_2^*}(\psi_2^* - \psi_1^*)\right] + \left(\beta^* - \frac{2f_0^2\Delta^*}{g''H_2^*}\right)\psi_2^* &= 0,
\end{align*}
\]

where the subscripts 1 and 2 denote the upper and lower layers, respectively, and the asterisk emphasizes that a quantity is in dimensional form. Here \( U^* \) and \( H^* \) are the zonal wind and layer thickness of the basic state, \( f_0^* \), \( \beta^* \), and \( g'' \) are the Coriolis parameter, beta, and reduced gravity, respectively. Note that shear \( \Delta^* \) is defined as \( U_1^* = U_2^* + 2\Delta^* \), and a normal-mode solution of the form \( \psi^* \approx \exp(ik^*(x^* - c^*t^*) + il^*y^*) \) is assumed for streamfunction. We define external length scales and the thickness ratio as:

\[
\begin{align*}
H_0^* &\equiv \frac{H_1^* + H_2^*}{2}; & L_D^2 &\equiv \frac{g''H_0^*}{2f_0^2}; & \delta &\equiv \frac{H_2^*}{H_0^*}; & \epsilon &\equiv 1 - \delta.
\end{align*}
\]

Note \( 0 < \delta < 2 \) and \( -1 < \epsilon < 1 \). Now nondimensionalize \( U_1^*, U_2^* \), and \( c^* \) by \( \Delta^* \), \( k^2 \) and \( l^2 \) by \( L_D^{-2} \), and \( \beta^* \) by \( \Delta^*L_D^{-2} \). Then Eqs. (1ab) become
\[
(\hat{u} + 1) \left[ -(1 + \varepsilon)(k^2 + l^2)\psi_1 + \frac{1}{2}(\psi_2 - \psi_1) \right] + [(1 + \varepsilon)\beta + 1]\psi_1 = 0,
\]
\[
(\hat{u} - 1) \left[ -(1 - \varepsilon)(k^2 + l^2)\psi_2 + \frac{1}{2}(\psi_2 - \psi_1) \right] + [(1 - \varepsilon)\beta - 1]\psi_2 = 0,
\]
where \(\hat{u} \equiv [(U_1 + U_2)/2] - c\) and the asterisk has been dropped. The dispersion relation is
\[
\mu(\mu + 1)\hat{u}^2 - [2\mu(p - \varepsilon) + p]\hat{u} - \mu^2 + \mu + p(p - \varepsilon) = 0,
\]
where \(\mu \equiv (1 - \varepsilon^2)(k^2 + l^2)\) and \(p \equiv \beta(1 - \varepsilon^2)\). The solution is unstable when
\[
\beta^{-1} > 1 - \varepsilon = \delta,
\]
which demonstrates the proportional relationship between the critical shear and the lower layer thickness (\(\beta^{-1}\) is routinely referred to as the criticality parameter).
Marginal stability is found at
\[
k^2 + l^2 = \frac{1 + \varepsilon\beta}{2(1 - \varepsilon^2)} \pm \sqrt{\frac{1 + \beta(1 + \varepsilon)[1 - \beta(1 - \varepsilon)]}{2(1 - \varepsilon^2)}},
\]
corresponding to short- and long-wave cutoffs. For \(\beta = 0\) the long-wave cutoff coincides with \(k = l = 0\) whereas the shortwave cutoff wavenumber is
\[
k^2 + l^2 = (1 - \varepsilon^2)^{-1/2} = [\delta(2 - \delta)]^{-1/2}.
\]
In this case the shortwave cutoff wavenumber minimizes at \(k^2 + l^2 = 1\) when \(\delta = 1\) (equal layer thickness), whereas it becomes infinite as \(\delta \to 0\) or 2 (i.e., when the thickness of either layer vanishes). When \(\beta^{-1} > \delta\) (i.e., supercritical) the growth rate is given by
\[
|kc| = |k\hat{u}| = \left[ k^2 - \mu^4 + \mu^2(1 - \varepsilon^2)(1 + \varepsilon\beta) - \beta^2(1 - \varepsilon^2)^2 / 4 \right]^{1/2} / \mu^2(\mu + 1)^2.
\]
These results have been widely known under different notations (Pedlosky 1987, Vallis 2006). To see how \(\varepsilon\) affects the maximum growth rate, it is instructive to consider the transverse mode \([l = 0, k^2 = \mu/(1 - \varepsilon^2)]\). Upon substitution (7) may be arranged into
\[
(kc_i)^2 = \frac{\mu (1 - \mu)}{1 + \mu} \left[ \frac{\varepsilon \mu^2 - \beta (1 - \mu^2)/2}{(1 - \mu^2) \mu (\mu + 1)^2} \right].
\]  
(8)

Since the last term on the right-hand side is nonpositive and the first term depends only on \( \mu \), the maximum growth rate occurs for a value of \( \mu \) that maximizes the first term and a combination of \( \varepsilon \) and \( \beta \) that makes the last term vanish for this \( \mu \):

\[
k c_{\max} = \sqrt{2} - 1, \quad \mu_{\max} = \sqrt{2} - 1, \quad \varepsilon_{\max} (\sqrt{2} - 1)^2 - \beta (1 - \mu_{\max}^2)/2 = 0.
\]  
(9)

From (9) the optimal thickness ratio is given by

\[
\varepsilon_{\max} = -\frac{(\sqrt{2} - 1)^2}{\beta} \sqrt{1 + \frac{(\sqrt{2} - 1)^4}{\beta^2}} \quad \text{or} \quad \delta_{\max} = 1 + \frac{(\sqrt{2} - 1)^2}{\beta} - \sqrt{1 + \frac{(\sqrt{2} - 1)^4}{\beta^2}}.
\]  
(10)

The other root of \( \varepsilon \) is negative and unsuitable for \( \beta > 0 \) [(9)]. Alternatively, by substituting \( \varepsilon_{\max} = 1 - \delta_{\max} \) in (9) and assuming \( \delta_{\max} < 2 \), an approximate expression

\[
\delta_{\max} \approx \frac{(\sqrt{2} - 1)^2}{(\sqrt{2} - 1)^2 + \beta} = \frac{1}{1 + 5.83 \beta}
\]  
(11)

may be obtained. Note \( \delta_{\max} \to 1 \) as \( \beta \to 0 \): the optimal lower-layer thickness for baroclinic instability approaches the mid-height of the model (equal layer thickness) in the limit of small beta (strong shear). In the opposite limit (large beta/weak shear) \( \delta_{\max} \propto \beta^{-1} \); namely the optimal thickness becomes inversely proportional to beta (proportional to shear). Once \( \varepsilon_{\max} \) (and \( \delta_{\max} \)) is obtained, the wavenumber \( k \) that maximizes the growth rate may be computed from (9) and (11) as

\[
k_{\max} = \left[ \mu_{\max} / (\delta_{\max} (2 - \delta_{\max})) \right]^{1/2} \approx 0.644 + 3.75 \beta / \sqrt{1 + 11.7 \beta}.
\]  
(12)

Without meridional shear in the basic state, the vertically integrated zonal-mean eddy PV flux vanishes (Pedlosky 1987 §7). With the flux written as the product of PV
gradient and eddy diffusivity $K$, this may be written as

$$K_1[(2-D)\beta+1]+K_2[\delta\beta-1]=0.$$  \hfill (13)

From (13) and (11), the diffusivity ratio $K_2/K_1$ that corresponds to $\delta_{\text{max}}$ is

$$(K_2/K_1)_{\text{max}} \approx 1+\beta\left[2.41+0.416/(1+4.83\beta)\right].$$  \hfill (14)

Thus $K_2 > K_1$ for $\beta > 0$ and the ratio increases with increasing $\beta$.

Figure 1 shows the growth rate of the transverse mode as a function of $k$ and $\delta$ for $\beta = 0$, 0.5, and 1.0. With $\beta = 0$, the growth rate maximizes at $k_{\text{max}} = 0.644$, $\delta_{\text{max}} = 1$ and it is symmetric about $\delta = 1$. There is no long-wave cutoff, whereas the shortwave cutoff wavenumber increases toward $\delta = 0$ and 2 as (6) predicts (Fig.1a). A positive $\beta$ breaks the symmetry and the growth rate maximum shifts to smaller $\delta$ and larger $k$, although the value of the maximum growth rate does not change. A long-wave cutoff is also introduced (Fig.1b). With $\beta = 1$, the flow is unstable only for $\delta < 1$. This corresponds to the criticality condition ($\beta^{-1} > \delta$, Fig.1c). The area of unstable domain becomes substantially smaller, with further shifts in $\delta_{\text{max}}$ and $k_{\text{max}}$.

Figure 2 plots $\delta_{\text{max}}$, $k_{\text{max}}$, and $(K_2/K_1)_{\text{max}}$ as functions of $\beta$, using both exact and approximate formulae. As $\beta$ increases $\delta_{\text{max}}$ decreases quickly, whereas the corresponding increase in $k_{\text{max}}$ is more gradual. The diffusivity ratio $(K_2/K_1)_{\text{max}}$ also increases with $\beta$, indicating that eddying motion is increasingly confined to the lower layer.

3. Discussion

Interpreting $\delta$ as the mode’s vertical scale allows one to draw analogy between Phillips’s and Charney’s (1947) models. For example, unstable normal modes in
Charney’s model do not have shortwave cutoff because they can adjust their vertical scale (critical level) as the wavelength decreases (Bretherton 1966). In Phillips’s model short waves are stabilized when their vertical scale falls below the prescribed \( \delta \).

However, if \( \delta \) is allowed to vary with the horizontal wavenumber, the Phillips model too can trace the most unstable mode to an arbitrarily large wavenumber. Figure 3 shows the maximum growth rate of the transverse mode for given \( \delta \) (here \( \beta = 1 \) is assumed) as a function of the corresponding wavenumber \( k_m(\delta) \). The growth rate resembles that of the Charney mode (Pierrehumbert and Swanson 1995 Fig3a): there is no shortwave cutoff.

One might also compare (11) with the vertical scale of the most unstable mode in Charney’s model, \( h^* \):

\[
\frac{h^*}{H^*} = \frac{1}{1 + (\beta^* N^* H^*)/(\Lambda^* f_0^2)} = \frac{1}{1 + (\beta^* L^2_D)/(\Lambda^* H^*)},
\]

(15)

where \( H^* \) is scale height, \( N^* \) is Brunt-Väisälä frequency and \( \Lambda^* \) is vertical shear (Held 1978). They are similar in that the mode depth is proportional to shear in the limit of weak shear/strong beta and constant in the opposite limit (mid-height and scale height, respectively). However, while \( h^* \) is proportional to the horizontal scale of the Charney mode, it is the square root of \( \delta_{\text{max}} \) that is proportional to the horizontal scale of the unstable modes in Phillips’s model \([\text{(12)}]\). As a result, for modes with similar horizontal scales \( \delta_{\text{max}} \) is much smaller than \( h^*/H^* \). For example, let \( 2\Delta^* = \Lambda^* H^* = 40 \text{ ms}^{-1} \), \( \beta^* = 1.6 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1} \), \( L^*_D = 800 \text{ km} \); typical for the Earth’s midlatitude. This gives \( \beta \sim 0.51 \) in (11) so \( \delta_{\text{max}} \approx 0.25 \), whereas (15) predicts \( h^*/H^* = 0.8 \), although the zonal wavenumbers of the modes are comparable.

When one specifies \( \delta \) in the two-layer model, one effectively singles out the
vertical structure of eddies. How does this choice affect the result, and on what basis should one choose $\delta$ in an initial-value problem? Unless there is a compelling reason to choose $\delta$ on the basis of observed stratification (e.g., there is a prevailing pycnocline at certain altitude) one sees two possible choices: (i) $\delta \approx \beta^{-1}$ and (ii) $\delta \approx \delta_{\max}$. In Phillips’s model when $\beta \neq 0$, tall modes are generally less unstable because shear’s negative contribution to the lower-layer PV gradient is diluted. All modes taller than $\delta_{c} = \beta^{-1}$ are stable as shear becomes subcritical (Fig.1c). The critical value $\delta_{c}$ may be interpreted as the depth of heat-transporting eddy required to neutralize the flow (Lindzen and Farrell 1980, Stone 1978). $\beta^{-1}$ evaluated from the observed shear suggests $\delta_{c} \geq 1$, i.e., to maintain the atmosphere marginally critical one needs deep eddies (Green 1970, Held 1982). Then, $\delta \approx \beta^{-1} \approx 1$ will be a reasonable choice if one aims to simulate a marginally critical state of an Earth-like atmosphere. However, such state is preconditioned to deep, slowly growing eddies and a priori eliminates shallower, fast growing eddies that could emerge if $\delta$ were smaller. For the set of parameters introduced earlier, $\delta = 1$ gives a maximum growth rate 0.332 at $k = 0.74$ and $K_{2} / K_{1} = 3.1$, whereas a much greater growth rate 0.414 is achieved if we choose $\delta = \delta_{\max} = 0.25$ at $k_{\max} = 0.97$ and $K_{2} / K_{1} = 2.2$. (Note that the equal layer thickness produces a greater asymmetry in diffusivity). Thus, by choosing $\delta = 0.25$ one would render the flow significantly more unstable and decrease the horizontal scale of eddy.

In Figs. 4 and 5 we compare two forced dissipative simulations of a turbulent baroclinic jet on a wide beta-plane channel using a two-layer model similar to Esler (2008). The mean shear is relaxed toward a jet (gray curves in Fig.5c,d), while the lower-
layer vorticity is damped by Ekman friction. The two runs are identical except $\delta = 1$ in the first and $\delta = 0.25$ (close to the optimal thickness at the axis of the jet) in the second, and the Ekman damping in the lower layer is four times stronger in the latter. The model is run for 900 days (the time series of eddy kinetic energy is shown in Fig.4) and statistics are constructed from the last 400 days of the simulations. Although the equilibrium wind profile is identical, with $\delta = 1$ the corresponding lower-layer PV gradient is negative only near the center of the channel (gray dashed curve in Fig.5a), whereas with $\delta = 0.25$ it is everywhere negative and strongly so near the axis of the jet (Fig.5b). In statistical steady state, the negative PV gradient in the former is nearly eliminated, whereas the lower-layer gradient in the latter, though substantially reduced, still remains strongly negative (Fig.5a,b). The mean shear in the statistical steady state with $\delta = 0.25$ is much weaker than that with $\delta = 1$ (Fig.5c,d), whereas eddy kinetic energy is higher (Fig.4), thus the conversion of available potential energy to kinetic energy is more efficient.

Since the two-layer model prescribes the vertical structure of baroclinic eddies with a preset thickness ratio, any single realization of statistical steady state should be interpreted with care; our results demonstrate that changing this parameter with all other parameters fixed can deliver a range of climate states, from marginally critical to strongly supercritical. Typically a supercritical state emerges when the width of unstable baroclinic zone is much wider than the horizontal scale of eddies (Pavan and Held 1996, Nakamura 1999, Jansen and Ferrari 2012). In the two-layer model, this condition is readily met under Earth-like parameters when the lower layer thickness is significantly less than the mid-height. By considering only equal layer thicknesses one could miss out a large class of climate states with high eddy activity that may be equally realizable.
References


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FIG. 2. Properties of the most unstable mode in Phillips’s model as function of $\beta$. (a) $\delta_{\text{max}}$, (b) $k_{\text{max}}$, (c) $(K_2 / K_1)_{\text{max}}$. Solid curves: exact solutions. Dashed curves: approximate solutions based on (11). See text for details.

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