Local finite-amplitude wave activity as a diagnostic
of anomalous weather events

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ABSTRACT

Finite-amplitude Rossby wave activity (FAWA) proposed by Nakamura and Zhu measures the waviness of quasigeostrophic potential vorticity (PV) contours and the associated modification of the zonal-mean zonal circulation but it does not distinguish longitudinally localized weather anomalies such as atmospheric blocking. In this article FAWA is generalized to local wave activity (LWA) to diagnose eddy-mean flow interaction on the regional scale. LWA quantifies longitude-by-longitude contributions to FAWA following the meridional displacement of PV from the circle of equivalent latitude. The zonal average of LWA recovers FAWA. The budget of LWA is governed by the zonal advection of LWA and the radiation stress of Rossby waves. The utility of the diagnostic is tested with barotropic vorticity equation on a sphere and meteorological reanalysis data. Compared with the previously derived Eulerian Impulse-Casimir wave activity, LWA tends to be less filamentary and emphasizes large isolated vortices involving reversals of meridional gradient of potential vorticity. A pronounced Northern Hemisphere blocking episode in late October 2012 is well captured by a high-amplitude, near stationary LWA. These analyses reveal that the nonacceleration relation holds approximately over regional scales: the growth of phase-averaged LWA and the deceleration of local zonal wind are highly correlated. However, marked departure from the exact nonacceleration relation is also observed during the analyzed blocking event, suggesting that the contributions from nonadiabatic processes to the blocking development are significant.
1. Introduction

Waves play an important role of rearranging angular momentum in the atmosphere. This process is summarized by the generalized Eliassen-Palm (E-P) relation (Andrews and McIntyre 1976)

\[
\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{F} = D + \mathcal{O}(\alpha^3),
\]

where \( A \) is the density of wave activity (negative angular pseudomomentum); \( t \) is time; \( \mathbf{F} \) is the generalized E-P flux, which represents radiation stress of the wave and equals the group velocity times the wave activity density for a slowly modulated, small-amplitude wave; \( D \) denotes nonconservative effects on wave activity; and \( \mathcal{O}(\alpha^3) \) represents terms of third (and higher) order of \( \alpha \), a measure of wave amplitude. For a small-amplitude, conservative wave the right-hand side terms are negligible and wave activity density changes only where there is nonzero E-P flux divergence.

The E-P flux divergence in turn drives the angular momentum of the mean flow, thus acting as the agent of wave-mean flow interaction.

Nakamura and Zhu (2010) (NZ10 hereafter) extended (1) for finite-amplitude Rossby waves and balanced eddies by introducing finite-amplitude wave activity (FAWA) based on the meridional displacement of quasigeostrophic potential vorticity (PV) from zonal symmetry. The formalism eliminates the cubic term from the right-hand side of (1) and extends the nonacceleration theorem (Charney and Drazin 1961) for an arbitrary eddy amplitude. This allows one to quantify the amount of the mean flow modification by the eddy (Nakamura and Solomon 2010, 2011).

Despite its amenability to data, FAWA is a zonally averaged quantity and incapable of distinguishing longitudinally isolated events such as atmospheric blocking. In this article we shall address this shortcoming by introducing local finite-amplitude wave activity (LWA). In essence, LWA quantifies longitude-by-longitude contributions to FAWA and as such recovers FAWA upon zonal averaging. As a first step into this topic, the present article concerns primarily the conser-
vative dynamics of local eddy-mean flow interaction. Explicit representation of nonconservative
dynamics (such as local diffusive flux of PV) will be deferred to a subsequent work. However,
when observed data deviates from the theory, it may be readily interpreted as an indication of non-
conservative effects. The material is organized as follows: section 2 lays out the theory. Section 3
demonstrates the utility of LWA using idealized simulations with barotropic vorticity equation on
a sphere. We will compare LWA with one of the existing local metrics of wave activity, Impulse-
Casimir wave activity (Killworth and McIntyre 1985; McIntyre and Shepherd 1987; Haynes 1988).
As an application of LWA, a blocking episode that steered superstorm Sandy to the East Coast of
the US in 2012 will be studied in section 4. Discussion and concluding remarks will follow in
section 5.

2. Theory

a. Finite-amplitude wave activity (FAWA)

In NZ10 finite-amplitude wave activity (FAWA) $A^\ast(y,z,t)$ is defined for latitude $y$, pressure
pseudoheight $z$ and time $t$ in terms of surface integrals of quasigeostrophic PV over two domains
of equal area, $D_1$ and $D_2$ (see Fig.1a). On a given $z$-surface, $D_1$ is enclosed by a wavy PV contour
of value $Q$, whereas $D_2$ is enclosed by a latitude circle at $y$. The requirement that the areas of $D_1$
and $D_2$ be identical enforces a one-to-one relation between $Q$ and $y$ for a given $z$. The latitude
$y(Q,z)$ that encloses the same area as the PV contour of value $Q$ is termed equivalent latitude
(Butchart and Remsberg 1986; Allen and Nakamura 2003). Then FAWA is given by

$$A^\ast(y,z,t) = \frac{1}{L_x} \left( \iint_{D_1} q(x,y',z,t) \, dx \, dy' - \iint_{D_2} q(x,y',z,t) \, dx \, dy' \right),$$  \hspace{1cm} (2)

$$D_1: \ 0 \leq x < L_x, \ q \geq Q(y,z); \hspace{1cm} D_2: \ 0 \leq x < L_x, \ y' \geq y(Q,z),$$
where $L_x$ is the length of the zonal circle. Note that under conservative quasigeostrophic dynamics $Q$-$y$ relation on the $z$ surface $Q(y,z)$ is independent of time because the wind that advects PV is divergence-free and thus area preserving. In (2) PV is defined as (Nakamura and Zhu 2010; Nakamura and Solomon 2010).

$$q(x,y,z,t) = \zeta + f \left[ 1 + e^{z/H} \frac{\partial}{\partial z} \left( e^{-z/H} \frac{(\theta - \bar{\theta}(z))}{d\bar{\theta}/dz} \right) \right], \quad \text{(3)}$$

where $f(y)$ is the Coriolis parameter, $\zeta$ is relative vorticity, $\theta$ is potential temperature, $\bar{\theta}(z)$ is its global horizontal average, and $H$ is a constant scale height. Now $q \geq Q$ everywhere in $D_1$, but $D_2$ includes some regions outside $D_1$ in which $q \leq Q$ and excludes parts of $D_1$ (Fig.1a), so the first integral in (2) is greater than the second, implying $A^* \geq 0$. The equal sign is achieved only when the PV contour coincides with the latitude circle. As noted by NZ10, in the small-amplitude, conservative limit $A^*(y,z,t)$ converges to the more familiar expression for wave activity, i.e. $A_s = \frac{1}{2} \frac{n^2}{\partial q/\partial y}$, where $\bar{(...)}$ denotes zonal average and $(...)'$ is the departure from it.

There are several advantages of $A^*$ over $A_s$. First, while $A_s$ obeys a nonacceleration theorem that is accurate only up to second order in wave amplitude, $O(\alpha^2)$, $A^*$ obeys an exact nonacceleration theorem under conservative dynamics. For example, for a frictionless barotropic flow ($\partial/\partial z = 0$)

$$\frac{\partial}{\partial t} (\bar{u} + A^*) = 0. \quad \text{(4)}$$

This is due to the fact that the first integral in (2) reduces to Kelvin’s circulation around a material contour of absolute vorticity while the second integral is (2$\pi$ times) the absolute angular momentum of the zonal-mean flow (see NZ10 for details). Equation (4) captures the essence of conservative eddy-mean flow interaction: acceleration of the zonal-mean flow is achieved at the expense of wave activity. NZ10 also derives the evolution equation of $A^*$ in the presence of diffusion and forcing [their (24a)]. Second, the PV-equivalent latitude relation $Q(y,z)$ may be exploited to define a zonally symmetric, time-invariant reference state. It is a hypothetical distribution of
PV that arises from ‘zonalizing’ the wavy PV contours on the $z$-surface without changing the enclosed areas. The corresponding flow and temperature field $u_{\text{REF}}(y,z), \theta_{\text{REF}}(y,z)$ may be inverted from $Q(y,z)$ assuming the quasigeostrophic dynamics: for barotropic case simply $u_{\text{REF}} = \bar{u} + A^\ast$.

The notion of reference state may be generalized to a ‘slowly varying state’ under nonconservative dynamics (Nakamura and Solomon 2010, 2011; Methven and Berrisford 2015). Third, unlike $A_s$ which turns negative when $\frac{\partial \bar{q}}{\partial y} < 0$, $A^\ast$ remains positive definite even when PV contours are overturned or fragmented since $\frac{\partial Q}{\partial y}$ is always positive by construction (Solomon and Nakamura 2012). $A^\ast$ is thus a more immediate measure of meridional displacement of a contour from zonal symmetry.

b. New formalism: the local finite-amplitude wave activity (LWA)

1) Definitions

Although the FAWA formalism quantifies waviness in the PV contours and the associated mean flow modification [see for example Solomon (2014) for stratospheric wave activity events], it is not suited to distinguish the longitudinal location of an isolated large-amplitude event such as blocking. To achieve this, $A^\ast(y,z,t)$ needs to be generalized to a function of longitude as well. In the following we assume that it is only the eddy properties that vary in longitude and continue to use $Q(y,z), u_{\text{REF}}(y,z)$, and $\theta_{\text{REF}}(y,z)$ as a reference state to define the eddy fields. A zonally symmetric reference state may not reflect the zonally asymmetric nature of the time-mean flow, but it is a required construct for the conservation of wave activity. Keep in mind that despite the enforced zonal symmetry, the reference state shares the same PV-area relation $Q(y,z)$ with the full wavy state so it is strongly constrained to the actual climate state (Nakamura and Solomon 2010, 2011; Methven and Berrisford 2015).
Because of the waviness in the flow, the PV contour of value $Q$ is displaced locally from $(x, y, z)$ to $(x, y + \eta(x, y, z, t), z)$, where $\eta(x, y, z, t)$ is defined positive northward. (As we will see below, $\eta$ can be multivalued in $y$.) Now let $0 \leq y' \leq \eta$ or $0 \geq y' \geq \eta$ depending on the sign of $\eta$. The eddy field is defined between $(x, y, z)$ and $(x, y + \eta, z)$ as

$$u_e(x, y + y', z, t) \equiv u(x, y + y', z, t) - u_{\text{REF}}(y, z), \quad (5)$$

$$v_e(x, y + y', z, t) \equiv v(x, y + y', z, t), \quad (6)$$

$$\theta_e(x, y + y', z, t) \equiv \theta(x, y + y', z, t) - \theta_{\text{REF}}(y, z), \quad (7)$$

$$q_e(x, y + y', z, t) \equiv q(x, y + y', z, t) - Q(y, z). \quad (8)$$

Notice that the displacement coordinate $y'$ is independent of $y$; in other words the eddy field is not defined globally as the total field minus the reference state but it needs to be redefined for each $y$.

By definition

$$q(x, y + \eta(x, y, z, t), z, t) = Q(y, z), \quad q_e(x, y + \eta(x, y, z, t), z, t) = 0 \quad (9)$$

and

$$0 = \frac{1}{L_x} \left( \iint_{D_1} dx dy' - \iint_{D_2} dx dy' \right) \Rightarrow \frac{1}{L_x} \int_0^{L_x} \left( \int_0^{\eta(x, y, z, t)} dy' \right) dx = \bar{\eta} = 0. \quad (10)$$

The definition of local finite-amplitude wave activity (LWA), $\tilde{A}^*(x, y, z, t)$, follows most naturally by rewriting (2) as:

$$A^*(y, z, t) = \frac{1}{L_x} \int_0^{L_x} \left( \int_{y + \eta(x, y, z, t)}^{y_{\text{max}}} q(x, y', z, t) dy' - \int_y^{y_{\text{max}}} q(x, y', z, t) dy' \right) dx$$
\[
\begin{align*}
&= -\frac{1}{L_x} \int_0^{L_x} \left( \int_y^{y+\eta} q(x,y',z,t) \, dy' \right) \, dx \\
&= -\frac{1}{L_x} \int_0^{L_x} \left( \int_0^{\eta+\eta} (q_e(x,y+y',z,t) + Q(y,z)) \, dy' \right) \, dx \\
&= \frac{1}{L_x} \int_0^{L_x} (\tilde{A}^*(x,y,z,t) - \eta Q(y,z)) \, dx \\
&= \tilde{A}^*(x,y,z,t) - \eta Q(y,z) \\
&= \tilde{A}^*(x,y,z,t),
\end{align*}
\]

where the last line used (10). We define LWA, \(\tilde{A}^*(x,y,z,t)\), as

\[
\tilde{A}^*(x,y,z,t) \equiv -\int_0^{\eta(x,y,z,t)} q_e(x,y+y',z,t) \, dy'
\]

or equivalently

\[
\tilde{A}^*(x,y,z,t) \equiv \int_{W_-} q_e(x,y+y',z,t) \, dy' - \int_{W_+} q_e(x,y+y',z,t) \, dy'
\]

\[W_+: \ 0 \leq y' \leq \eta_+(x,y,z,t), \ q \leq Q(y,z); \quad W_-: \ 0 \geq y' \geq \eta_-(x,y,z,t), \ q \geq Q(y,z).\]

In the above we use \((\ldots)^*\) to denote wave activity that is a function of both longitude and latitude.

It is evident from (11) that the zonal average of LWA recovers FAWA (2). In practice LWA is computed by evaluating (13) and (14). When there are multiple crossings of the PV contour with the meridian at a given \(x\), we take the furthest crossings from equivalent latitude as \(\eta_+ > 0\) and \(\eta_- < 0\) in (14) and use the PV constraint in (14) to sample the correct segments along the integral path. (Numerically this amounts to a conditional box counting along the meridian.) Computation of \(\tilde{A}^*\) is illustrated in Fig.1b. On a given \(z\)-surface, PV is generally greater on the northern side of the wavy contour than on the southern side, such that \(q_e \geq 0\) in the red lobes and \(q_e \leq 0\) in the blue lobes. The line integral of \(q_e\) over the red area and minus the line integral of \(q_e\) over the blue are both positive, which makes \(\tilde{A}^*\) a positive definite quantity. By construction, \(\tilde{A}^*(x,y,z,t)\)
is Lagrangian (nonlocal) in $y$ and Eularian (local) in $x$. Notice that since LWA vanishes at the nodes, (i.e., crossing of the PV contour and equivalent latitude) it contains the phase structure of the waves in addition to the amplitude. In the small-amplitude, conservative limit (12) becomes

$$\tilde{A}^*(x,y,z,t) \rightarrow \tilde{A}_s^* = \frac{1}{2} \frac{q'^2}{\partial q/\partial y}. \quad (15)$$

2) LOCAL WAVE ACTIVITY AND PV GRADIENT

NZ10 shows in their Eqn. (18) that FAWA bridges the Lagrangian- and Eularian-mean PV via

$$\frac{\partial A^*}{\partial y}(y,z,t) = \bar{q}(y,z,t) - Q(y,z). \quad (16)$$

Analogous result may be obtained for LWA when differentiating (12) with respect to $y$ (see Appendix A for the derivation):

$$\frac{\partial \tilde{A}^*}{\partial y}(x,y,z,t) = q(x,y,z,t) - Q(y,z) + \frac{\partial Q}{\partial y} \eta(x,y,z,t). \quad (17)$$

When $\eta$ is multivalued, the sum of all values is used. Zonally averaging (17) and using (10) recovers (16). Differentiating this with respect to $y$ again yields

$$\frac{\partial q}{\partial y} = \frac{\partial Q}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\partial \tilde{A}^*}{\partial y} - \frac{\partial Q}{\partial y} \eta \right), \quad (18)$$

which generalizes the relation (19) in NZ10. Thus the criterion for local reversal of PV gradient is

$$\frac{\partial Q}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\partial \tilde{A}^*}{\partial y} - \frac{\partial Q}{\partial y} \eta \right) < 0. \quad (19)$$

Polvani and Plumb (1992) discuss two regimes of wave breaking in the context of vortex dynamics: major Rossby wave breaking that disrupts the vortex dynamics and microbreaking that only sheds filaments and does not affect the vortex significantly. [See also Dritschel (1988).] In terms of LWA, a major breaking would satisfy (19) as well as a large amplification in LWA $\Delta \tilde{A}^* \approx u$, whereas microbreaking would satisfy (19) without significant changes in $\tilde{A}^*$. 

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3) LOCAL WAVE ACTIVITY BUDGET

The governing equation for LWA may be obtained by taking the time derivative of (12) (see Appendix B for the derivation):

$$\frac{\partial \tilde{A}^*}{\partial t} = -e^{-z/H} \nabla \cdot (F_{adv} + F_{EP}),$$

(20)

where

$$F_{adv} \equiv e^{-z/H} \left( u_{REF}(y,z) \tilde{A}^* - \int_0^\eta (u_e q_e)dy', 0, 0 \right)$$

(21)

denotes the advective flux of LWA, whereas

$$F_{EP} \equiv e^{-z/H} \left( \frac{1}{2} \left( v_e^2 - u_e^2 - \frac{R}{H} \frac{e^{-\kappa z/H} \theta_e^2}{d\theta/dz} \right), -u_e v_e, \frac{fv_e \theta_e}{d\theta/dz} \right)$$

(22)

is the generalized E-P flux (Plumb 1985). Here $\kappa = R/c_p$, $R$ is gas constant, and $c_p$ is specific heat at constant pressure. The first term in the $x$-component of (21) is of $\mathcal{O}(\eta^2)$ at small amplitude and converges to $e^{-z/H} \bar{u} \tilde{A}^*$. The second term, which is of $\mathcal{O}(\eta^3)$ and represents the Stokes drift flux of $\tilde{A}^*$, only becomes significant at finite amplitude. In this article, no further effort will be made to remove phase information from $\tilde{A}^*$ and fluxes other than averaging over a longitudinal-window. [The methods described in Plumb (1985) and Takaya and Nakamura (2001) are not readily applicable to finite-amplitude wave activity.]

c. Relationship to Impulse-Casimir wave activity

Another well-known measure of local wave activity is Impulse-Casimir wave activity (ICWA), first introduced by Killworth and McIntyre (1985) and further developed by McIntyre and Shepherd (1987) and Haynes (1988). ICWA may be defined with respect to any zonally uniform, time-independent reference state in which PV ($q_0$) is a monotonic function of $y$. It is defined as

$$A_{IC}(x,y,z,t) = \int_0^{q^*(x,y,z,t)} (Y(q_0 + \hat{q}^*) - Y(q_0)) d\hat{q}^*,$$

(23)
where \( q(x,y,z,t) \equiv q^*(x,y,z,t) + q_0(y,z) \), and \( Y(q_0,z) \) is an inverse function of \( q_0(y,z) \) for a given \( z \). \( A_{IC}(x,y,z,t) \) obeys (Killworth and McIntyre 1985; Haynes 1988)

\[
\frac{\partial A_{IC}}{\partial t} = -e^{z/H} \nabla \cdot (F_{\text{adv}}^* + F_{\text{EP}}^*) ,
\]

where \( F_{\text{adv}}^* \equiv e^{-z/H} \left( (u_0(y,z) + u^*) A_{IC}, \ v^* A_{IC}, \ 0 \right) \),

\[
F_{\text{EP}}^* \equiv e^{-z/H} \left( \frac{1}{2} \left( v^{*2} - u^{*2} - \frac{R e^{-\kappa z/H} \theta^{*2}}{d\theta/dz} \right), \ -u^* v^*, \ \frac{f v^* \theta^*}{d\theta/dz} \right),
\]

where the asterisk denotes the local departure from the reference state. If \( q_0 \) is chosen to be identical with \( Q(y,z) \), there is a close relationship between \( \tilde{A}^* \) and \( A_{IC} \). As illustrated in Fig.2, on the \( y-q \) plane \( \tilde{A}^*(x,y_1,z,t) \) is given by the area bounded by \( q = Q(y_1,z), y = y_1 \) and the curve \( q = q(x,y,z,t) \) (Fig.2a), whereas \( A_{IC}(x,y_1,z,t) \) is given by the area bounded by \( q = q(x,y_1,z,t), y = y_1 \) and the curve \( q = Q(y,z) \) (Fig.2b). When the eddy is of small-amplitude (i.e. \( q(x,y,z,t) \approx Q(y,z) \)) these two areas are similar and both converge to (15). At where \( q(x,y,z,t) = Q(y,z) \) (nodes), they both vanish. However, once the PV gradient \( \frac{\partial q}{\partial y} \) is reversed, \( \tilde{A}^* \) becomes positive even at \( q = Q(y,z) \) (Fig.2c), whereas \( A_{IC} \) remains zero (Fig.2d). In fact, \( \tilde{A}^* \) tends to be greatest around the gradient reversal because both red and blue lobes in Fig.1b (\( x = x_3 \)) contribute to it. Consequently, \( \tilde{A}^* \) emphasizes the region of wave breaking more than \( A_{IC} \) does, as we will see in the next section.

Both wave activities obey similar equations [(20) and (24)] but while the ICWA equation is written entirely in terms of Eulerian quantities, the LWA equation involves line integrals and hence Lagrangian in the meridional. A crucial difference arising from this is an extra meridional advection term \( \frac{\partial}{\partial y} (v^* A_{IC}) \) in (25) which does not have a counterpart in (21). The meridional advection of \( \tilde{A}^* \) is absorbed in the movement of PV contour and does not appear in (21). The extra term in (25) prevents \( A_{IC} \) from possessing an exact nonacceleration theorem (NZ10).
The nonacceleration relation (4) shows conservation of the sum of zonal-mean zonal wind and wave activity in a frictionless barotropic flow, but it does not tell whether the deceleration of the zonal-mean wind is due to growth of a localized wave packet or simultaneous growth of multiple wave packets over longitudes. To understand the dynamics of a localized phenomenon such as blocking, it is desirable to characterize eddy-mean flow interaction over a regional scale.

To formulate local eddy-mean flow interaction in a form analogous to (4), we start by taking the density weighted vertical average of (20):

$$\frac{\partial}{\partial t} \left\langle \tilde{A}^* \right\rangle = - \frac{\partial}{\partial x} \left( \left\langle u_{REF} \tilde{A}^* \right\rangle - \left\langle \int_0^\eta (u_e q_e) dy' \right\rangle + \frac{\left\langle v_e^2 \right\rangle - \left\langle u_e^2 \right\rangle}{2} - R \frac{e^{-\kappa z/H}}{\theta_e^2} \frac{d \tilde{\theta}}{dz} \right\rangle$$

$$- \frac{\partial}{\partial y} \left( - \left\langle u_e v_e \right\rangle \right) + f v_e \theta_e H \frac{d \tilde{\theta}}{dz} \bigg|_{z=0},$$

(27)

where the angle bracket denotes the density weighted vertical average

$$\left\langle \cdots \right\rangle \equiv \frac{\int_0^\infty (\cdots) e^{-z/H} dz}{\int_0^\infty e^{-z/H} dz} = \frac{\int_0^\infty (\cdots) e^{-z/H} dz}{H}.$$  

(28)

As will be shown in section 4, because of the density weighting this column average mainly samples the troposphere. The corresponding vertically averaged zonal momentum equation is

$$\frac{\partial}{\partial t} (u) = - \frac{\partial}{\partial x} \left( u_{REF} u_e + u_e^2 \right) - \frac{\partial}{\partial y} (u_e v_e) + f \left( v_e \right) - \frac{\partial}{\partial x} (\Phi_e),$$

(29)

where $\Phi_e$ is the eddy geopotential. We also introduce local surface wave activity $\tilde{B}^*$

$$\tilde{B}^*(x, y, t) \equiv - \frac{f}{H d \tilde{\theta}} \int_0^{\eta(x, y, t)} \theta_e (x, y + y', t) dy' \bigg|_{z=0},$$

(30)

which is analogous to (12) but defined based on the meridional displacement of surface potential temperature contour. Note by definition $\tilde{B}^* \leq 0$ and its zonal average recovers the surface FAWA (NZ10, Wang and Nakamura (2015), submitted to GRL). $\tilde{B}^*$ obeys the equation

$$\frac{\partial \tilde{B}^*}{\partial t} = - \frac{\partial}{\partial x} \left( u_{REF} (y, 0) \tilde{B}^* - \frac{f}{H d \tilde{\theta}} \int_0^{\eta(x, y, t)} (u_e \theta_e) dy' \bigg|_{z=0} \right) - \frac{f v_e \theta_e}{H d \tilde{\theta}} \bigg|_{z=0}.$$  

(31)
Adding (27), (29), and (31) one obtains

\[ \frac{\partial}{\partial t} \left( \langle u \rangle + \langle \tilde{A}^* \rangle + \tilde{B}^* \right) \]

\[ = -\frac{\partial}{\partial x} \left( \langle u_{\text{REF}}(u + \tilde{A}^*) \rangle + u_{\text{REF}}(y, 0)\tilde{B}^* - \left( \int_0^\eta (u_e q_e) dy' \right) \right) - \frac{f}{H d \tilde{\theta} / dz} \int_0^{\eta(x,y,t)} (u_e \theta_e) dy' \bigg|_{z=0} \]

\[ + \frac{1}{2} \left( u_e^2 + v_e^2 \right) - \frac{R}{2H} \left( e^{-\kappa z/H} \frac{d \tilde{\theta}}{dz} \right) + f \psi_e - \langle \Phi_e \rangle , \quad (32) \]

where \( \psi_e \) is barotropic streamfunction such that \( \langle v_e \rangle = \frac{\partial \psi_e}{\partial x} \). Notice the zonal average of (32) gives

\[ \frac{\partial}{\partial t} \left( \langle \bar{u} \rangle + \langle \tilde{A}^* \rangle + \bar{B}^* \right) = 0, \quad (33) \]

a baroclinic extension of (4). Now define \textit{regional average} over a longitudinal window of \( \Delta x \), denoted by \([\ldots]_{\Delta x}\)

\[ [g(x,y,t)]_{\Delta x} \equiv \frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} g(x,y,t) dx. \quad (34) \]

Averaging (32) over \( \Delta x \) would give

\[ \frac{\partial}{\partial t} \left[ \langle u \rangle + \langle \tilde{A}^* \rangle + \tilde{B}^* \right]_{\Delta x} = -\frac{1}{\Delta x} \left( \ldots \right) \bigg|_{x-\Delta x/2}^{x+\Delta x/2} . \quad (35) \]

If the atmospheric wave packets satisfy the Wentzel-Kramers-Brillouin (WKB) approximation [Bühler (2014) Ch.2] such that the wavelength is much smaller than the length-scale of the packet, by choosing \( \Delta x \) to be the wavelength, the right-hand side of (35) would be a small residual due to the slow modulation of wave properties in \( x \). Thus on short timescales

\[ \frac{\partial}{\partial t} \left[ \langle u \rangle + \langle \tilde{A}^* \rangle + \tilde{B}^* \right]_{\Delta x} \approx 0. \quad (36) \]

This is the approximate local nonacceleration theorem in the WKB sense: the sum of the phase-averaged barotropic LWA, surface LWA, and zonal wind remains unchanged in the conservative limit. If this is the case, growth of wave amplitude occurs at the expense of local zonal wind and vice versa. A migratory wave tends to slow down as it grows in amplitude because it decelerates the local westerly wind and weakens zonal advection. Furthermore, positive feedback might
arise because the locally weakened westerly will arrest and accumulate more LWA from upstream, leading to even more deceleration of the flow (Swanson 2000). Note that a corresponding nonacceleration theorem does not hold for $A_{IC}$ because of the additional meridional flux term in (24). In the next section, we will compare LWA and ICWA using idealized numerical simulations in which finite-amplitude Rossby waves are allowed to interact with shear flow on a rotating sphere. The extent to which the above conservation law for LWA is satisfied will be examined. In section 4, the LWA diagnostic is applied to meteorological reanalysis data to identify and analyze atmospheric blocking events.

3. Numerical experiment

Experimental setup

The utility of the LWA diagnostic will be tested in a barotropic decay simulation of finite-amplitude Rossby waves as described by Held and Phillipps (1987) (hereafter HP87). The governing barotropic vorticity equation reads:

$$\frac{\partial \zeta}{\partial t} + J(\psi, f + \zeta) = -\nu \nabla^4 \zeta,$$

(37)

where $f = 2\Omega \sin \phi$ is the Coriolis parameter ($\Omega$ is the rotation rate of the sphere), $\zeta$ is relative vorticity, $J$ is Jacobian, $\psi$ is streamfunction and $\nu$ is hyperviscosity, which we choose to damp the shortest resolved wave by a factor of $1/48$ daily. We impose an initial zonal-mean flow as prescribed by HP87

$$\bar{u}(\phi, t = 0) = 25 \cos \phi - 30 \cos^3 \phi + 300 \sin^2 \phi \cos^6 \phi,$$

(38)

which mimics the zonal-mean wind in the upper troposphere with westerlies in the midlatitudes and an easterly at the equator. We also impose a vorticity anomaly $\zeta'$ of the form in HP87, which is a Gaussian wave packet in meridional centered at $\phi_m = 45^\circ$ with zonal wavenumber $m = 6$. 

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Then another wave with \((m,n) = (4,6)\) is added to break the zonal symmetry and allows merging of wave packets. Here \(m\) and \(n\) are the zonal and total wavenumbers, respectively. The explicit form of \(\zeta'\) (see Fig.3, top) is

\[
\zeta' = \zeta_0 \cos \phi e^{-\left(\frac{\phi - \phi_m}{\sigma}\right)^2} \cos 6\lambda + \zeta_1 (\sin^2 \phi - 1)^2 (11 \sin^2 \phi - 1) \cos 4\lambda, 
\]  

(39)

where \(\phi\) is latitude, \(\lambda\) is longitude, \(\zeta_0 = 8 \times 10^{-5} \text{s}^{-1}\), \(\zeta_1 = 9 \times 10^{-6} \text{s}^{-1}\), \(\phi_m = 45^\circ\) and \(\sigma = 10^\circ\). We discretize the equation with a standard spectral transform method truncated at T170 on a Gaussian grid of resolution \(512 \times 256\). The Adams-Bashforth third-order scheme (see Durran (2013) Ch.2.4) is used to integrate the equation with a time increment of \(\Delta t = 360\text{s}\) until the major wave packet decays completely. The computation of \(\tilde{A}^*\) and \(A_{IC}\) is implemented on instantaneous snapshots of vorticity field obtained from the simulation. Since the model is barotropic, the third dimension in the fluxes (21), (22), (25) and (26) is ignored, and potential temperature and surface LWA \(\tilde{B}^*\) are set to zero. The local nonacceleration relation (36) is simplified to

\[
\frac{\partial}{\partial t} [u + \tilde{A}^*]_{\Delta x} \approx 0.
\]  

(40)

**Comparison between \(\tilde{A}^*\) and \(A_{IC}\)**

The overall flow evolution is similar to that in HP87: the wave packet initially located on the north side of the jet axis splits into poleward- and equatorward migrating tracks, and as they approach critical lines at the flanks of the jet they produce wave breaking. The initial vorticity pattern consists of six pairs of positive and negative anomalies (Fig.3, top), but their strengths are not symmetric due to the addition of small-amplitude, secondary wave \((m,n) = (4,6)\). As the wave packet begins to separate meridionally, six positive vorticity anomalies move northward whereas six negative anomalies move southward, and by day 3 the vorticity contours begin to overturn at
the flanks of the jet. (Here, anomalies are defined as departures from the zonal mean of the initial state. See Fig.3, bottom.)

The snapshots of absolute vorticity, LWA ($\tilde{A}^*$) and ICWA ($A_{IC}$), are shown for day 3 and 6 in Fig.4 over the Northern Hemisphere. The positive anomalies form isolated vortices around 50$^\circ$, whereas the negative anomalies develop marked anticyclonic tilt at the equatorward flank of the jet (Fig.4, top left). Both $\tilde{A}^*$ and $A_{IC}$ identify large vorticity anomalies but there are substantial differences between their spatial distribution. $\tilde{A}^*$ emphasizes the three largest positive anomalies, although they are shifted and elongated poleward from the actual locations of the vorticity anomalies (Fig.4, center left). This is a nonlocal effect of $\tilde{A}^*$: the isolated vortices are indeed associated with a higher equivalent latitude. $A_{IC}$ also picks up the isolated vortices but they tend to be much more compact and intense than $\tilde{A}^*$. Also, the structure of $A_{IC}$ around the negative anomalies appears more filamentary than $\tilde{A}^*$. Part of this difference is due, as explained in the previous section (in Fig.2d), to the fact that $A_{IC}$ tends to suppress wave amplitude in the region of reversed vorticity gradient: for example, the value of $A_{IC}$ drops from a maximum to zero to the north and south of isolated vortices. By day 6 (Fig.4, right), a pair of vortices start to merge poleward of the jet around 10$^\circ$ – 110$^\circ$ and 190$^\circ$ – 290$^\circ$E. In the $\tilde{A}^*$ plot, the merging vortices appear as one bulk structure, whereas in $A_{IC}$ they are more fragmented. On the equatorward flank of the jet, wave breaking causes the negative vorticity anomalies to roll up. The $\tilde{A}^*$ captures these emerging vortices faithfully but $A_{IC}$ is highly filamentary around them. Similar filamentary structures of $A_{IC}$ have been observed in previous analyses related to baroclinic life cycles and Rossby wave breaking (Magnusdottir and Haynes 1996; Thuburn and Lagneau 1999).
Local negative correlation between $\tilde{A}^*(x,y,t)$ and $u(x,y,t)$

For a zonal-mean state, the nonacceleration relation (4) describes conservative eddy-mean flow interaction: $\tilde{u}$ accelerates at the expense of $A^*$ and vice versa, thus their variation is antiphase. $u_{\text{REF}} \equiv \bar{u} + A^*$ is constant in time if the dynamics is conservative, so any changes in $u_{\text{REF}}$ are due to nonconservative processes; in the present case, they represent damping of FAWA through vorticity mixing (enstrophy dissipation by hyperviscosity). Since the initial condition (39) creates interference of zonal wavenumbers 4 and 6, the resultant flow has a zonal periodicity $\pi$. We expect $[\chi(x,y,t)]\pi$ [cf. (34)] for any physical quantity $\chi$ to be identical with the zonal mean. The question is whether the nonacceleration relation holds at a more regional scale $\Delta x < \pi$ as in (36). Although there is no strict periodicity below $\pi$ due to the presence of multiple waves, $m = 6$ still remains a dominant zonal wavenumber so $\Delta x = \pi/3$ would be a reasonable choice of the averaging window.

The values of $[u(x,y,t)]_{\Delta x}$ and $[\tilde{A}^*(x,y,t)]_{\Delta x}$ are computed between $x = 0^\circ$ and $60^\circ E$ at $30^\circ N$ and plotted as functions of time in the top panel of Fig.5. This particular latitude is chosen because a prominent wave breaking occurs around here (Fig.4).

The opposite tendency of the two quantities is evident, particularly during the early stage of simulation. Also plotted in the top panel are the sum $[u + \tilde{A}^*]_{\Delta x}$ and $u_{\text{REF}}(y,t)$. The zonal averages of the two quantities are identical. The slow variation of $u_{\text{REF}}$ reflects rearrangement of angular momentum by vorticity mixing, which is not included in (4). $[u + \tilde{A}^*]_{\Delta x}$ follows $u_{\text{REF}}$ generally well, suggesting that the long-term changes in $[u + \tilde{A}^*]_{\Delta x}$ are due to mixing. The disagreements are largely due to periodic modulation of $[\tilde{A}^*]_{\Delta x}$ by waves with wavelengths greater than $\pi/3$, but the range of fluctuation in $[u + \tilde{A}^*]_{\Delta x}$ is generally smaller than that of $[u]_{\Delta x}$ or $[\tilde{A}^*]_{\Delta x}$ alone, attesting to the overall validity of (36). Similar analysis is performed for $[A_{IC}]_{\Delta x}$ in the bottom panel of Fig.5. Compared to $[\tilde{A}^*]_{\Delta x}$, $[A_{IC}]_{\Delta x}$ varies much less, and its anticorrelation with $[u]_{\Delta x}$ is far less evident.
Accordingly, the sum of $[A_{IC}]_{\Delta x}$ and $[u]_{\Delta x}$ varies more in time. This demonstrates that the local nonacceleration relation (36) is generally not applicable to $A_{IC}$.

Figure 6 extends the above analysis to the entire latitude circle by showing the longitude-time (Hovmöller) cross sections (Hovmöller 1949) of $[u]_{\Delta x}$, $[\tilde{A}^*]_{\Delta x}$, and $[u + \tilde{A}^*]_{\Delta x}$ anomalies (departure from the time mean) at $30^\circ$N ($\Delta x = \pi/3$). Because of the averaging the fields are devoid of zonal wavenumber 6, the predominant structure in the unfiltered data. Instead, the analysis picks out the emerging wavenumber 2, which modulates the averaged quantities. The negative correlation between $[u]_{\Delta x}$ and $[\tilde{A}^*]_{\Delta x}$ is again evident, and it holds not only in time but also in longitude (particularly strong in the early stage). This is important because it suggests that the nonacceleration relation (40) is applicable regionally. On the other hand, (40) is not perfect: $[u + \tilde{A}^*]_{\Delta x}$ shows significant residual in the bottom panel. As mentioned above, it is partly due to nonconservative effects (vorticity mixing). It also contains a wavenumber 2 component, which represents ‘group propagation’ of $[u + \tilde{A}^*]_{\Delta x}$ expressed by the right-hand side of (35). Although the amplitude of this variation is smaller than the amplitude of $u$ or $\tilde{A}^*$, its non-negligible magnitude suggests that the scale separation required for (40) is insufficient. (In the present case the wavelength of the dominant wave is $\pi/3$ whereas the packet size is $\pi$.)

We have repeated the analysis varying $\Delta x$ ($\pi/6$ and $2\pi/3$) and found (not surprisingly) that $[u + \tilde{A}^*]_{\Delta x}$ deviates from $u_{\text{REF}}$ more when we reduce $\Delta x$ further. Arguably this simulation is a special case in which the wave spectra are highly discrete. In a sense it is even less obvious how best to choose an optimal $\Delta x$ when the waves have broader spectra. We will see in the next section that dealing with the real atmospheric data, horizontal averaging may actually be forgone.
4. Analysis of a blocking episode

Blocking is a phenomenon at midlatitudes in which a large-scale pressure anomaly remains stationary. The normal westerly winds in the mid- to upper troposphere are diverted meridionally along the blocking pattern and the wind within the block is often replaced by easterlies. Lejenäs and Økland (1983) observed that blocking occurs at longitudes where the latitudinal average of the zonal wind at 500 hPa is easterly. Tibaldi and Molteni (1990) added an additional requirement that the average wind be westerly poleward of the block. Such description of blocking based on reversal of zonal wind is a kinematic statement. Given the potential of (36) to quantify the slowing down of the flow by finite-amplitude eddies, the formalism is well suited for identifying and investigating blocking events with meteorological data.

In this section, we explore the extent to which the dynamics of a real blocking episode may be characterized based on the conservation relation (36). In particular, we will study the blocking episode that steered the superstorm Sandy to the East Coast of US during October 2012 with the LWA formalism. The interior- and surface LWA as well as the zonal wind are evaluated from the European Centre for Medium-Range Weather Forecasts ERA-Interim reanalysis product (Dee et al. 2011) at a horizontal resolution of $1.5^\circ \times 1.5^\circ$.

First, we evaluate PV from (3) on 49 equally spaced pressure pseudoheight as described in Nakamura and Solomon (2010) (we assume $H = 7 km$). Then, we compute $\tilde{A}^*$ from (13). $\tilde{B}^*$ is computed from (30) except we have replaced the surface potential temperature with the potential temperature at $866 hPa$ to avoid the nonquasigeostrophic effects in the boundary layer.

Overview of zonal wind and LWA in Northern Autumn 2012

Longitude-time (Hovmöller) diagrams for the barotropic components of zonal wind $\langle u \rangle$, LWA $\langle \tilde{A}^* \rangle + \tilde{B}^*$ and their sum at $42^\circ N$ during this season are shown in Fig.7. Notice that in this analysis
we are not using any horizontal average defined by (34). Indeed, the prevalent streak pattern in LWA (second panel) suggests an average eastward migration of LWA at about 11ms$^{-1}$, consistent with the phase speed of baroclinic waves (Williams and Colucci 2010). Thus, we believe that the streak pattern in LWA largely reflects the phase structure. However, the eastward migration of LWA is occasionally interrupted by large-amplitude, quasistationary features. A close correspondence is observed between these large LWA events and the reversal of zonal wind (i.e. negative $\langle u \rangle$) in the left panel, although the magnitude of fluctuation in LWA is about twice as large as that of the zonal wind (notice the different color scales for the two quantities). The fluctuation of their sum (Fig. 7, right column) has a smaller variation than $\langle \tilde{A}^* \rangle + \tilde{B}^*$. The simultaneous growth of LWA and the deceleration of zonal flow are characteristic of blocking. Remarkably neither does the unfiltered phase signal hinder the detection of blocking nor does removal of the phase by averaging improve the result significantly. It appears that LWA has no problem detecting the packet structure of blocks without regional averaging (34). Part of the reason is that the last two terms in (32) nearly cancel in geostrophic balance and that the vertical averaging in the other right-hand side terms, when the phase surfaces are tilted vertically, achieves the same effect as the phase averaging.

**Blocking episode around North American East Coast during 27 Oct - 2 Nov 2012**

Now we focus on a single blocking episode that occurred during Oct 27-Nov 2 over the North Atlantic. (The longitudinal range of concern is marked by the black lines in Fig.7.) This episode was characterized by a persistent blocking pattern in the mid- to upper troposphere and contributed to the steering of superstorm Sandy at right angle to the East Coast of US (Blake et al. 2013). Figures 8 and 9 respectively show PV and the corresponding LWA ($\tilde{A}^*$) at 240 hPa. There is an intrusion of low PV air poleward at 290$^\circ$E and 40$^\circ$N which remains stagnant longitudinally.
(relative to other eastward migrating features) for 2 days and eventually split into two asymmetric vortices. The smaller vortex that moves westward accompanied Sandy in-shore. The location and magnitude of the block are well-captured by high values of $\bar{A}^*$ in Fig. 9.

One might ask how the barotropic component (density-weighted vertical average) samples the vertical distribution of LWA associated with blocking. Figure 10 shows the vertical structure of $\bar{A}^*$ (left) and density weighted LWA ($e^{-z/H}\bar{A}^*$, right). Even though the pattern of blocking is apparent in $\bar{A}^*$ only at the upper levels (i.e. 300-150hPa), density weighting indeed brings out a vertically coherent structure of high LWA as shown in Fig.10 (right). Thus, what we observe in Fig.7 represents a persistent block affecting an entire troposphere and not just upper levels, both in terms of the accumulation of LWA and the deceleration of the flow.

To examine the extent to which the local nonacceleration relation accounts for the simultaneous accumulation of LWA and deceleration of zonal flow, in Fig.11 we show $\Delta \langle u \rangle$ (red), $\Delta(\langle \bar{A}^* \rangle + \bar{B}^*)$ (blue), and their sum (green), averaged longitudinally over $270^\circ - 330^\circ$E (the longitudes bounded by the black lines in 7) at different latitudes within the meridional extent of the blocking episode. Here $\Delta$ denotes departure from the seasonal average. This graph is analogous to Fig.5 for the barotropic decay simulation. The correlation coefficient of the time series of $\langle u \rangle$ and $\langle \bar{A}^* \rangle + \bar{B}^*$ throughout the analysis period is displayed at the top left hand corners of each plot.

There are several remarkable features from the plots. First, there is a strong negative correlation in the time series of $\Delta \langle u \rangle$ (red) and $\Delta(\langle \bar{A}^* \rangle + \bar{B}^*)$ (blue), clearly indicating the antiphase covariation of the two quantities expected from the nonacceleration relation. This relation is particularly visible during the block (27 Oct - 2 Nov) when the amplitude of the wave is large. Second, the $60^\circ$-longitudinal average of $\Delta(\langle u \rangle + \langle \bar{A}^* \rangle + \bar{B}^*)$ (green) weakly oscillates about zero except during the time of blocking formation when LWA grows large. Its peak value exceeds $20ms^{-1}$. Since (36) states that this quantity is approximately invariant in time under conservative dynamics, it
suggests that conservative dynamics cannot fully account for the occurrence of blocking. Given that the deceleration of the zonal flow has only half of the magnitude of the LWA anomaly, diabatic heating or other nonconservative processes are necessary to fuel the remainder of LWA anomaly associated with this block. The discrepancy does not depend strongly on the averaging window, suggesting that the violation of the WKB condition is not the primary cause of the deviation from (36).

5. Summary and discussion

We have generalized the notion of FAWA introduced by NZ10 to LWA, a diagnostic for longitudinally localized wave events, and tested its utility in both a barotropic model and meteorological data. A significant advantage of LWA over the existing wave activity measures is that it carries over the nonacceleration relation (36) of FAWA to regional scales, albeit within the WKB approximation. This explicitly attributes local deceleration of the zonal flow to accumulation of wave activity.

A robust negative correlation is found between $\langle u \rangle$ and $\langle \tilde{A}^* \rangle + \tilde{B}^*$ in both a simulated wave breaking and an observed blocking event, suggesting that the quasiadiabatic eddy-mean flow interaction is indeed of leading order importance in these weather events. Nevertheless, the variation of $\langle \tilde{A}^* \rangle$ during the blocking event is about twice as much as that of $\langle u \rangle$, which implies that not all LWA growth is accounted for by the simultaneous deceleration of the zonal flow. Diabatic and other nonconservative processes are responsible for half the budget of $\tilde{A}^*$ anomaly. This perspective is consistent with a recent study based on the formalism by NZ10 (Wang and Nakamura 2015, submitted to GRL) that shows that the variability of eddy-driven jet in austral summer is largely dictated by conservative dynamics of wave-mean flow interaction but moderated by strong thermal damping of surface wave activity. Strong damping of (negative) $\tilde{B}^*$ would render the values and
variability of $\langle \tilde{A}^* \rangle + \tilde{B}^*$ higher than those expected under the adiabatic condition, consistent with our analysis. The precise role of nonadiabatic effects on blocking formation will be a subject of subsequent work.

LWA dynamically connects the two criteria of blocking indices: (1) deceleration or even reversal of westerlies (Lejenäs and Økland 1983; Tibaldi and Molteni 1990) and (2) large amplitude of anomalies or gradient reversal in either geopotential height (at 500hPa) (Barnes et al. 2012; Dunn-Sigouin and Son 2013) or potential temperature on constant potential vorticity surface (2PVU) (Pelly and Hoskins 2003). Hoskins et al. (1985) suggests that meridional gradient reversal of potential temperature on a constant PV surface could imply a reversal of westerlies via the invertibility principle, but such a relation is not explicit. LWA can potentially serve as a blocking index because a large LWA will automatically leads to a significant deceleration of local zonal wind, to the extent that nonacceleration relation holds.

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APPENDIX A

Derivation of Eqn. (17)

Starting from (12)
\[ \tilde{A}^*(x, y, z, t) = - \int_0^{y+\eta(x,y,z,t)} q_e(x, y + y', z, t) \, dy' \]
\[ = - \int_y^{y+\eta(x,y,z,t)} \left[ q(x, y', z, t) - Q(y, z) \right] \, dy' \]
\[ = - \int_y^{y+\eta(x,y,z,t)} q(x, y', z, t) \, dy' + Q(y, z) \eta(x, y, z, t). \] (A1)

Then by taking the derivative with respect to \( y \) and using the Leibniz rule and (9)
\[ \frac{\partial \tilde{A}^*}{\partial y} = - \left( 1 + \frac{\partial \eta}{\partial y} \right) q(x, y + \eta, z, t) + q(x, y, z, t) + \frac{\partial Q}{\partial y} \eta + Q(y, z) \frac{\partial \eta}{\partial y} \]
\[ = - \left( 1 + \frac{\partial \eta}{\partial y} \right) Q(y, z) + q(x, y, z, t) + \frac{\partial Q}{\partial y} \eta + Q(y, z) \frac{\partial \eta}{\partial y} \]
\[ = q(x, y, z, t) - Q(y, z) + \frac{\partial Q}{\partial y} \eta(x, y, z, t). \] (A2)

This is (17).

APPENDIX B

Derivation of Eqns. (20)–(22)

From (12)
\[ \tilde{A}^*(x, y, z, t) \equiv - \int_0^{\eta(x,y,z,t)} q_e(x, y + y', z, t) \, dy' \]
\[ = - \int_0^{\eta(x,y,z,t)} \left( q(x, y + y', z, t) - Q(y, z) \right) \, dy'. \] (B1)

Taking the time derivative and using the Leibniz rule and (9)
\[ \frac{\partial}{\partial t} \tilde{A}^*(x, y, z, t) = - \frac{\partial \eta}{\partial t} q_e(x, y + \eta, z) - \int_0^{\eta} \frac{\partial q(x, y + y', z, t)}{\partial t} \, dy' - \int_0^{\eta} \frac{\partial q(x, y + y', z, t)}{\partial t} \, dy'. \] (B2)
Conservation of PV in \((x,y')\) is

\[
0 = \frac{dq}{dt} = \frac{\partial q}{\partial t} + (u_{\text{REF}}(y, z) + u_e) \frac{\partial q_e}{\partial x} + v_e \frac{\partial}{\partial y'} (q_e + Q(y, z))
\]

\[
= \frac{\partial q}{\partial t} + u_{\text{REF}}(y, z) \frac{\partial q_e}{\partial x} + \frac{\partial}{\partial x} (u_e q_e) + \frac{\partial}{\partial y'} (v_e q_e).
\]  

(B3)

Note that the spatial derivative of the eddy quantities is taken with respect to the coordinates \((x,y')\), and we used nondivergence of \((u_e, v_e)\). Substituting (B3) into (B2) yields [with repeated use of the Leibniz rule and (9)]

\[
\frac{\partial}{\partial t} \tilde{A}^*(x, y, z, t) = \int_0^\eta \left( u_{\text{REF}}(y, z) \frac{\partial q_e}{\partial x} + \frac{\partial (u_e q_e)}{\partial x} + \frac{\partial (v_e q_e)}{\partial y'} \right) dy'
\]

\[
= u_{\text{REF}} \int_0^\eta \frac{\partial q_e}{\partial x} dy' + \int_0^\eta \frac{\partial (u_e q_e)}{\partial x} dy' + (v_e q_e)_{y'=0} = (v_e q_e)_{y'=0}
\]

\[
= u_{\text{REF}} \frac{\partial}{\partial x} \int_0^\eta q_e dy' + \frac{\partial}{\partial x} \int_0^\eta u_e q_e dy' - (v_e q_e)_{y'=0}
\]

\[
- \frac{\partial \eta}{\partial x} (u_{\text{REF}}(y, z) + u_e(x + \eta + z)) q_e(x + \eta + z)
\]

\[
= -u_{\text{REF}} \frac{\partial \tilde{A}^*}{\partial x} + \frac{\partial}{\partial x} \int_0^\eta u_e q_e dy' - (v_e q_e)_{y'=0}.
\]  

(B4)

Rewriting the last term with the Taylor identity and thermal wind balance relation,

\[
\frac{\partial}{\partial t} \tilde{A}^*(x, y, z) = -u_{\text{REF}} \frac{\partial \tilde{A}^*}{\partial x} + \frac{\partial}{\partial x} \int_0^\eta u_e q_e dy' - \frac{\partial}{\partial x} \left[ \frac{1}{2} \left( v_e^2 - u_e^2 - \frac{R}{H} \frac{e^{-z/H} \theta^2}{d\theta/dz} \right) \right]
\]

\[
+ \frac{\partial}{\partial y'} (u_e v_e) - e^{-z/H} \frac{\partial}{\partial z} \left( f e^{-z/H} v_e \theta_e \right).
\]  

(B5)

This is (20)–(22).
References


Fig. 1. (a) A schematic diagram showing (on the x-y plane) the surface integral domains $D_1$ and $D_2$ in Eqn. (2), the definition of finite-amplitude wave activity (FAWA) of Nakamura and Zhu (2010). The horizontal dashed lines indicate the equivalent latitude corresponding to the PV contour shown, such that the pink and blue areas are the same. (b) A schematic diagram illustrating how to compute the local finite-amplitude wave activity in Eqns. (13)-(14). The wavy curve indicates a contour of PV, above which the PV values are greater than below. Inside the red lobes $q_e \geq 0$ and inside the blue lobes $q_e \leq 0$. Four points are chosen to illustrate how the domain of integral is chosen. 

\[
\hat{A}^2(x_1,y) = - \int_{\Omega_1} q_e(x,y+y',z,t)dy', \quad \hat{A}^2(x_2,y) = \int_{\Omega_2} q_e(x,y+y',z,t)dy', \quad \hat{A}^2(x_3,y) = \int_{\Omega_3} q_e(x,y+y',z,t)dy', \quad \hat{A}^2(x_4,y) = \int_{\Omega_4} q_e(x,y+y',z,t)dy'.
\]

Fig. 2. Comparison of $\hat{A}^2$ and $A_{IC}$. Curves indicate latitudinal cross sections of PV at fixed x and z. Solid curves: $Q(y,z)$. Dashed curves: $q(x,y,z,t)$. (a) Shaded areas indicate $\hat{A}^2$ at $y = y_1$ and $y = y_2$. See Eqn. (12). (b) Same as (a) but for $A_{IC}$. See Eqn. (23). (c) $q(x,y,z,t)$ involves gradient reversal. Shaded areas indicate $\hat{A}^2$ at $y = y_3$. (d) Same as (c) but for $A_{IC}$. $A_{IC}$ at $y = y_3$ is zero.

Fig. 3. Top: Initial vorticity anomaly (39) for the barotropic decay experiment (contour interval: $8.25 \times 10^{-6} \text{s}^{-1}$; negative values are dashed). Bottom: Same as top but for the difference between the relative vorticity on Day 3 and the initial zonal-mean relative vorticity.

Fig. 4. Longitude-latitude distributions of absolute vorticity (top, unit $s^{-1}$), $\hat{A}^2$ (center, unit $ms^{-1}$) and $A_{IC}$ (bottom, $ms^{-1}$) from the barotropic decay experiment. Left column: day 3. Right column: day 6.

Fig. 5. Top: Evolution of $u$ (red) and $\hat{A}^2$ (blue), $(u + \hat{A}^2)$ (green) averaged over a fixed longitudinal window of 60° at 30° during the barotropic decay simulation. Also plotted is $u_{REF}$ (black). Bottom: same as top but for $A_{IC}$ (blue) and $u + A_{IC}$ (green). The unit of the vertical axis is $ms^{-1}$.

Fig. 6. Top: Longitude-time (Hovmöller) cross section of $u$ anomaly (departure from time-mean) at 30° during the barotropic decay simulation. At each instant the quantity is averaged over 60° window in longitude ($\Delta x = 60^\circ$ in (34)). Middle: same as top but for $\hat{A}^2$. Bottom: same as top but for $(u + \hat{A}^2)$. Unit: $ms^{-1}$.

Fig. 7. Hovmöller cross section of $\langle u \rangle$ (left), $\langle \hat{A}^2 \rangle + \hat{B}$ (middle) and $\langle u \rangle + \langle \hat{A}^2 \rangle + \hat{B}$ (right) at 42°N from September 1 to November 31, 2012. The vertical black lines bound the range of longitudes where the zonal average is taken to obtain the local nonacceleration relation. Unit: $ms^{-1}$.

Fig. 8. Evolution of quasigeostrophic PV at 240 hPa from 29 Oct 00:00 UTC to 30 Oct, 2012 12:00 UTC (with 12-hour interval). Unit: $s^{-1}$.

Fig. 9. Evolution of local wave activity ($\hat{A}^2$) at 240 hPa from 29 Oct 00:00 UTC to 30 Oct, 2012 12:00 UTC (with 12-hour interval). Unit: $ms^{-1}$.

Fig. 10. The vertical structures of (left) local wave activity ($\hat{A}^2$) computed with Eqn. (13) on each pressure surface and (right) density-weighted $\hat{A}^2$ on 29 Oct, 2012 at 18:00 UTC at 45°N.
The surface wave activity is obtained from data on the $p = 866hPa$ level. Unit: $ms^{-1}$. Note different color scales for the two panels.

**Fig. 11.** Evolution of the anomalies (departure from the seasonal mean) of $\langle u \rangle$ (red), $\langle \tilde{A}^* \rangle + \tilde{B}^*$ (blue), their sum (green) at various latitudes (marked at the top right corner) within $270^\circ - 330^\circ E$. The global zonal average of $\langle u \rangle + \langle \tilde{A}^* \rangle + \tilde{B}^*$ is also shown in black. The unit of the vertical axis is $ms^{-1}$. The correlation between the time profile of $\langle u \rangle$ and $\langle \tilde{A}^* \rangle + \tilde{B}^*$ are shown at the top left corner. See text for details.
Equivalent Latitude
\[ y = Y(Q) \]
Contour of PV = Q
Higher PV: \( q > Q(y), q_e > 0 \)
Lower PV: \( q < Q(y), q_e < 0 \)

(b) Local Finite-amplitude Wave Activity:

(a) Finite-amplitude Wave Activity (Nakamura and Zhu 2010):

Fig. 1. (a) A schematic diagram showing (on the x-y plane) the surface integral domains \( D_1 \) and \( D_2 \) in Eqn. (2), the definition of finite-amplitude wave activity (FAWA) of Nakamura and Zhu (2010). The horizontal dashed lines indicate the equivalent latitude corresponding to the PV contour shown, such that the pink and blue areas are the same. (b) A schematic diagram illustrating how to compute the local finite-amplitude wave activity in Eqns. (13)-(14). The wavy curve indicates a contour of PV, above which the PV values are greater than below. Inside the red lobes \( q_e \geq 0 \) and inside the blue lobes \( q_e \leq 0 \). Four points are chosen to illustrate how the domain of integral is chosen. 

\[
\tilde{A}^+(x_1, y) = \int_{W_{1+}} q_e(x,y+y',z,t)dy'; \quad \tilde{A}^-(x_2, y) = \int_{W_{2-}} q_e(x,y+y',z,t)dy'; \\
\tilde{A}^+(x_3, y) = \int_{W_{3+}} q_e(x,y+y',z,t)dy' - \int_{W_{3-}} q_e(x,y+y',z,t)dy'; \quad \tilde{A}^-(x_4, y) = \int_{W_{4-}} q_e(x,y+y',z,t)dy';
\]
FIG. 2. Comparison of $\tilde{A}^*$ and $A_{IC}$. Curves indicate latitudinal cross sections of PV at fixed $x$ and $z$. Solid curves: $Q(y,z)$. Dashed curves: $q(x,y,z,t)$. (a) Shaded areas indicate $\tilde{A}^*$ at $y = y_1$ and $y = y_2$. See Eqn. (12). (b) Same as (a) but for $A_{IC}$. See Eqn. (23). (c) $q(x,y,z,t)$ involves gradient reversal. Shaded areas indicate $\tilde{A}^*$ at $y = y_3$. (d) Same as (c) but for $A_{IC}$. $A_{IC}$ at $y = y_3$ is zero.
FIG. 3. Top: Initial vorticity anomaly (39) for the barotropic decay experiment (contour interval: 8.25 × 10⁻⁶ s⁻¹; negative values are dashed). Bottom: Same as top but for the difference between the relative vorticity on Day 3 and the initial zonal-mean relative vorticity.
FIG. 4. Longitude-latitude distributions of absolute vorticity (top, unit $s^{-1}$), $\tilde{A}^*$ (center, unit $m/s$) and $A_{IC}$ (bottom, $m/s$) from the barotropic decay experiment. Left column: day 3. Right column: day 6.
Fig. 5. Top: Evolution of $u$ (red) and $\bar{A}^*$ (blue), $(u + \bar{A}^*)$ (green) averaged over a fixed longitudinal window of 60° at 30° during the barotropic decay simulation. Also plotted is $u_{\text{REF}}$ (black). Bottom: same as top but for $A_{\text{IC}}$ (blue) and $u + A_{\text{IC}}$ (green). The unit of the vertical axis is $ms^{-1}$.

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FIG. 6. Top: Longitude-time (Hovmöller) cross section of $u$ anomaly (departure from time-mean) at $30^\circ$ during the barotropic decay simulation. At each instant the quantity is averaged over $60^\circ$ window in longitude ($\Delta x = 60^\circ$ in (34)). Middle: same as top but for $\tilde{A}^*$. Bottom: same as top but for $(u + \tilde{A}^*)$. Unit: $m s^{-1}$. 

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Fig. 7. Hovmöller cross section of $\langle u \rangle$ (left), $\langle \tilde{A}^* \rangle + \tilde{B}$ (middle) and $\langle u_g \rangle + \langle \tilde{A}^* \rangle + \tilde{B}$ (right) at 42°N from September 1 to November 31, 2012. The vertical black lines bound the range of longitudes where the zonal average is taken to obtain the local nonacceleration relation. Unit: $ms^{-1}$. 
Fig. 8. Evolution of quasigeostrophic PV at 240 hPa from 29 Oct 00:00 UTC to 30 Oct, 2012 12:00 UTC (with 12-hour interval). Unit: s\(^{-1}\).
Fig. 9. Evolution of local wave activity ($\tilde{A}^*$) at 240 hPa from 29 Oct 00:00 UTC to 30 Oct, 2012 12:00 UTC (with 12-hour interval). Unit: $ms^{-1}$. 
FIG. 10. The vertical structures of (left) local wave activity ($\tilde{A}^*$) computed with Eqn. (13) on each pressure surface and (right) density-weighted $\tilde{A}^*$ on 29 Oct, 2012 at 18:00 UTC at 45°N. The surface wave activity is obtained from data on the $p = 866hPa$ level. Unit: $ms^{-1}$. Note different color scales for the two panels.
FIG. 11. Evolution of the anomalies (departure from the seasonal mean) of $\langle u \rangle$ (red), $\langle \tilde{A}^* \rangle + \tilde{B}^*$ (blue), their sum (green) at various latitudes (marked at the top right corner) within $270^\circ - 330^\circ E$. The global zonal average of $\langle u \rangle + \langle \tilde{A}^* \rangle + \tilde{B}^*$ is also shown in black. The unit of the vertical axis is $ms^{-1}$. The correlation between the time profile of $\langle u \rangle$ and $\langle \tilde{A}^* \rangle + \tilde{B}^*$ are shown at the top left corner. See text for details.