An Exact Lagrangian-Mean Wave Activity for Finite-Amplitude Disturbances to Barotropic Flow on the Sphere

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The finite-amplitude wave activity introduced recently by Nakamura et al. measures disturbances in terms of the areal displacement of potential vorticity (PV) from zonal symmetry and allows calculation of the mean flow modification through a time-integrated nonacceleration theorem. This article investigates both theoretically and numerically how this wave activity relates to previously defined quantities such as the generalized Lagrangian mean (GLM) pseudomomentum and the impulse-Casimir (IC) wave activity in the context of barotropic flow on the sphere. It is shown that under the barotropic constraint both the new and GLM formalisms derive the nonacceleration theorem from the conservation of Kelvin’s circulation, but the two differ in the way the circulation is partitioned into a mean flow and wave activity/pseudomomentum. In the conservative small-amplitude limit, the new wave activity differs from the (negative of) GLM pseudomomentum by the impulse associated with the Stokes corrections to the mean position of the PV contour. The new and IC wave activities take an identical form in the small-amplitude limit, but a geometrical interpretation of the latter illustrates their departure at finite amplitude. A simple jet forcing experiment is used to examine the quantitative differences among these diagnostics. The new and the IC wave activities behave similarly in the domain average, but they differ substantially in the local profiles, the former being more closely related to the flow modification. Despite their close conceptual relationship, the GLM pseudomomentum fails to capture the salient features of the wave activity because the impulse associated with the Stokes correction to the PV contour dominates the former.

Key Words: wave activity, potential vorticity, Kelvin’s circulation, Lagrangian-mean, barotropic

1. Introduction

The bulk of fluid dynamics, both theory and modeling, has been developed in an Eulerian framework. Since the pioneering work of Reynolds (1895), eddies have typically been defined as the departure from some spatial or temporal average. If we define the

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Eulerian zonal mean of a scalar field $\chi(\lambda, \mu, t)$ on the surface of a sphere as

$$\chi(\mu,t) = \frac{1}{2\pi} \int_0^{2\pi} \chi(\lambda, \mu, t) d\lambda,$$

(1.1)

where $\lambda$ is longitude, $\mu = \sin \phi$, $\phi \in [-\pi/2, \pi/2]$ is latitude, and $t$ is time, then the eddy component is given by

$$\chi'(\lambda, \mu, t) = \chi(\lambda, \mu, t) - \chi(\mu, t).$$

(1.2)

This decomposition forms a basis for our understanding of wave properties and the interactions of eddies with mean flows in geophysical fluids. In particular the notion of pseudomomentum, the angular momentum carried by the eddies themselves, and its conservation properties are fundamental to our understanding of atmospheric circulation (e.g. Vallis (2006) §§7, 12). Linear pseudomomentum can be defined in terms of small wavy departures of vorticity $\omega$ from a time-independent, zonally uniform state $\omega^R$, $\zeta' \equiv \omega - \omega^R$.

(Here $\omega^R = \overline{\omega}$ but we use the subscript $R$ to emphasize the time independence.) For example, for barotropic flow on the sphere (e.g. Held & Phillips (1987))

$$A(\mu, t) = a \cos \phi \frac{\zeta'^2}{8} d\omega^R/d\mu.$$

(1.3)

The quantity $A$ is the negative of pseudomomentum density but we will call it simply wave activity, following the standard nomenclature. In the absence of friction and other nonconservative effects $A$ satisfies

$$\frac{\partial A}{\partial t} + \overline{v' \zeta'} = O(\alpha^3),$$

(1.4)

where $v$ is the meridional component of the flow velocity and $\alpha$ is a measure of eddy amplitude. The right-hand side includes the advection of $A$ by the eddies. Wave activity is related to the zonal-mean zonal velocity $\overline{u}$ through the second term, the poleward flux of vorticity by the eddies:

$$\frac{\partial \overline{u}}{\partial t} = v' \zeta' = - \frac{\partial A}{\partial t} + O(\alpha^3) \Rightarrow \frac{\partial}{\partial t} (\overline{u} + A) = O(\alpha^3).$$

(1.5)

Thus, in the conservative, small-amplitude limit the zonal-mean flow remains unchanged if wave activity is steady, a result accurate through $O(\alpha^2)$. This is a simplest form of the nonacceleration theorem due to Charney & Drazin (1961). In a balanced flow the eddy vorticity flux may be expressed as the convergence of eddy momentum flux through Taylor’s identity

$$\overline{v' \zeta'} = -(a \cos \phi)^{-1} \frac{\partial}{\partial \mu} \overline{u' v' (1 - \mu^2)},$$

(1.6)

where $a$ is the radius of the sphere. Therefore $\overline{v' \zeta' \cos \phi}$ vanishes upon integration over the domain, which, together with (1.5) leads to the conservation of the domain-integrated $A \cos \phi$ (accurate through $O(\alpha^2)$) and $\overline{\pi \cos \phi}$ (exact). The global conservation of $A \cos \phi$ is a key to formulating linear stability theorems (Kuo (1949), Rayleigh (1896 (reprinted 1945), Vallis (2006), §7).

An effort to extend (1.5) to a finite-amplitude regime while maintaining the Eulerian description of the flow is due to Killworth & McIntyre (1985), McIntyre & Shepherd (1987), and Haynes (1988). These authors construct a finite-amplitude wave activity that obeys an exact local flux conservation by combining Kelvin’s impulse and Casimir functions associated with absolute vorticity (or potential vorticity, PV). The impulse-Casimir
(IC) wave activity is measured relative to an arbitrary, zonally symmetric reference state that does not depend on time. The IC wave activity is $O(\alpha^2)$ as $\alpha \to 0$ and it converges to the linear wave activity (1.3) in this limit. The global conservation of IC wave activity allows one to derive finite-amplitude stability theorems for shear flows (McIntyre & Shepherd (1987), Shepherd (1988), Shepherd (1989)). However, the $O(\alpha^3)$ term in (1.5) becomes locally dominant at large amplitudes, undermining the role of wave activity as a local driver of the mean flow. Also, the definition of IC wave activity depends on the choice of the reference state, which is not \textit{a priori} obvious for the real atmosphere. For these reasons, the IC wave activity is rarely used for the diagnosis of the real atmospheric flows.

A significant conceptual advantage may be gained if one switches to a Lagrangian perspective. The generalized Lagrangian mean (GLM) theory (Andrews & McIntyre (1978b), Andrews & McIntyre (1978c), McIntyre (1980)) defines a mean state by averaging quantities over air parcels following their motions. For example, Lagrangian-mean properties of a wavy material tube displaced from a zonal circle may be characterized as a function of its center-of-mass latitude, $\phi_m$. In particular, the relationship between the GLM zonal momentum and pseudomomentum at $\phi_m$ is analogous to (1.5) yet \textit{exact}, that is, there is no residual on the right-hand side (Andrews & McIntyre (1978b); see also Eq. (2.16) below). This allows one to interpret pseudomomentum as the local driver of the mean flow even at finite amplitude. However, the application of the GLM diagnostic to data is considered impractical, as the material tubes can evolve into complex shapes when eddy amplitude is large and their center-of-mass latitudes may not sample the globe uniformly. As a result, the GLM diagnostic is routinely replaced by its Eulerian cousin, the transformed Eulerian mean (TEM), which approximates the Lagrangian-mean motion with the mean eddy fluxes evaluated at fixed latitude (Andrews & McIntyre (1976), Andrews & McIntyre (1978a)). In particular, the generalized Eliassen-Palm flux and its divergence (or the eddy vorticity flux in the above barotropic case) have been widely used as a key diagnostic for the tendency of wave activity (e.g., Edmon \textit{et al}. (1980)). See Bühler (2009) for a comprehensive review of the GLM theory.

Recently Nakamura \& Zhu (2010a), Nakamura \& Zhu (2010b), (hereafter referred to as NZ10ab), Nakamura \& Solomon (2010a) and Nakamura \& Solomon (2010b) (NS10ab hereafter) introduced a finite-amplitude wave activity based on the instantaneous areal displacement of PV contours from zonal symmetry. In the context of barotropic flow on the sphere,

$$A^*(\mu_e, t) = \frac{1}{2\pi a \cos \phi_e} \left( \iint_{\omega \geq \omega_e} \omega dS - \iint_{\mu \geq \mu_e} \omega dS \right),$$

(1.7)

where $dS = a^2 d\lambda d\mu$ is the area element, $\omega = 2\Omega\mu + \zeta$ is the absolute vorticity ($\Omega$ is the rotation rate of the sphere and $\zeta$ is relative vorticity), and $\omega_e$ is the value corresponding to \textit{equivalent latitude} $\mu_e$, i.e., the latitude of the $\omega$-contour in an \textit{eddy-free reference state} after $\omega$ is ‘zonalized’ without changing the enclosed area. Measuring the departures from a mean state which is independent of longitude, this finite-amplitude wave activity is most accurately described as the pseudomomentum defined in equivalent latitude coordinate. With the Stokes theorem the two integrals in (1.7) can be expressed as Kelvin’s circulation about the material vorticity contour ($C$) and the zonal mean circulation at a given equivalent latitude ($\overline{C}$)

$$A^*(\mu_e, t) = \frac{1}{2\pi a \cos(\phi_e)} (C(\mu_e, t) - \overline{C}(\mu_e, t)).$$

(1.8)
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It is easy to show (NZ10a, NS10a) that $A^* \geq 0$ for an arbitrary geometry of the $\omega$-contour and (e.g., appendix of NS10a) that the small-amplitude, conservative limit of $A^*$ equals $A$ if $\omega_R$ in (1.3) is taken as the absolute vorticity as a function of $\mu_e$, which is invariant with time. Furthermore, in the conservative (but finite-amplitude) limit, $A^*$ satisfies an exact nonacceleration theorem

$$\frac{\partial}{\partial t} (\bar{u} + A^*) = 0 \quad (1.9)$$

at each $\mu_e$. This result arises naturally if one applies Kelvin’s theorem to (1.8) and notes that $(2\pi a \cos \phi)^{-1} \bar{C} = \bar{u} + \Omega a \cos \phi$.

By integrating Eq. (1.9) with time from a hypothetical eddy-free reference state $(\bar{u}, A^*) = (\bar{u}_{\text{ref}}, 0)$ to the current observed state $(\bar{u}_{\text{obs}}, A_{\text{obs}}^*)$, one can quantify the adjustment to the mean flow, or the departure from the reference state, in terms of the observed wave activity

$$\Delta u \equiv \bar{u}_{\text{obs}} - \bar{u}_{\text{ref}} = -A_{\text{obs}} \quad (1.10)$$

which can be evaluated with instantaneous data using (1.7). The above formalism has been extended to baroclinic atmosphere and applied to reanalysis data to quantify the adiabatic adjustment in the general circulation of the atmosphere due to finite-amplitude eddies (NS10ab). Furthermore, (1.9) implies that the reference state flow

$$u_{\text{ref}} = \pi + A^* = \frac{1}{2\pi a \cos(\phi_e)} \left( \iint \omega dS - \iint 2\Omega \mu dS \right) \quad (1.11)$$

is time-independent in the conservative limit, and more generally, it responds only to forcing and dissipation and filters out advective eddy-mean flow interaction. Therefore, $u_{\text{ref}}$ is a useful diagnostic for the nonconservative effects on the mean flow. For example, NZ10ab use $u_{\text{ref}}$ to diagnose the effects of mixing on the sharpening of jets in $\beta$-plane turbulence. It has been demonstrated by NS10ab that (the baroclinic generalization of) $u_{\text{ref}}$ captures a greater seasonal variation and less high-frequency fluctuations than $\pi$, suggesting that $u_{\text{ref}}$ can exhibit better signal-to-noise ratio as a climate diagnostic.

Thus the new diagnostic carries a flavor of Lagrangian-mean formalism by incorporating the material displacement with the contours of absolute vorticity but avoids the difficulty of the GLM formalism by using equivalent latitude as a coordinate, which, unlike the center-of-mass latitude, is guaranteed to cover the entire globe uniformly even when the eddy amplitude is large. Moreover, as suggested by NZ10a, the global conservation of $A^* \cos \phi_e$ may be used to formulate the finite-amplitude (Lyapunov) stability theorems similar to those derived from the IC wave activity (McIntyre & Shepherd (1987), Shepherd (1988), Shepherd (1989)).

The purpose of this article is twofold. First, we shall theoretically relate $A^*$ with the GLM pseudomomentum and the IC wave activity by explicitly working out the formalisms for barotropic flow on the sphere. This will clarify, for example, why $A^*$, while being a close relative to the GLM pseudomomentum, avoids the difficulties of the GLM formulation and why $A^*$ and the IC wave activity diverge at finite amplitude. Second, we shall relate $A^*$ quantitatively with these more traditional measures of wave activity in a simple barotropic experiment on the sphere in which a jet is formed by localized wave forcing. This is a fundamental phenomenon of barotropic flow on a rotating sphere, investigated over the years analytically, experimentally and numerically (Kuo (1951), McEwan et al. (1980), Vallis (2006)). The transport by waves results in a convergence of eastward momentum into the forcing region and westward momentum elsewhere. This is similar to the mechanism which maintains the midlatitude westerly winds against surface
finite amplitude wave activity (Charney 1959, Eliassen & Palm 1961, Edmon et al. 1980, Boer & Shepherd 1983). Application of the different forms of wave activity to this controlled experiment clarifies the strengths and weaknesses of the diagnostics.

In the next section the connection with the GLM pseudomomentum will be examined using the small-amplitude theory. In §3 a geometrical interpretation of the impulse-Casimir method will be used to highlight the difference between the new and IC wave activities. In §4 the model and experimental design will be discussed briefly, followed by an analysis of the simulation with the various measures of wave activity. Sensitivity of these diagnostics to model truncation is also addressed. The final section provides a summary.

2. Circulation, wave activity, and Generalized Lagrangian Mean

In this section, we will examine the link between the new wave activity and the GLM pseudomomentum by constructing a small-amplitude theory of the latter for barotropic flow on the sphere. To avoid the difficulty described by McIntyre (1980 §4.2), we will not derive the GLM set from the full 3D problem, but apply the GLM theory directly to the 2D barotropic vorticity equation on the sphere. Since the conservation of Kelvin’s circulation is central to both the GLM theory and the novel construction of NZ10ab and NS10ab, we will first outline the forms of circulation as well as the methods employed for its calculation.

2.1. Circulation in the \((\lambda, \mu)\)-plane

Consider a conservative, barotropic fluid motion on a sphere of radius \(a\) with rotation rate \(\Omega\). Since the motion is independent of height, we will work in the longitude-sine latitude coordinate \((\lambda, \mu) = (\lambda, \sin \phi)\), where \(\phi\) is latitude. In these coordinates the area element of the fluid is given by

\[
dS = a^2 \cos \phi \, d\lambda d\phi = a^2 d\lambda d\mu,
\]

so the spherical surface is mapped onto a rectangular plane, which makes the kinematics similar to those in Cartesian coordinates. Note that the Eulerian fluid velocity is nondivergent

\[
V = (U, V) = (a\dot{\lambda}, a\dot{\mu}) \quad \text{and} \quad \partial U/\partial \lambda + \partial V/\partial \mu = 0,
\]

where \(U = u/\cos \phi\) and \(V = v \cos \phi\) and \((u, v)\) are the velocities in the \((\lambda, \phi)\) coordinates. The momentum equations in the \((\lambda, \mu)\)-plane are

\[
\frac{D}{Dt} \left[U(1 - \mu^2)\right] - 2\Omega \mu V = -\frac{1}{a} \frac{\partial p}{\partial \lambda},
\]

\[
\frac{D}{Dt} \left(\frac{V}{1 - \mu^2}\right) + 2\Omega \mu U + \frac{\mu}{a} \left(U^2 - \frac{V^2}{(1 - \mu^2)^2}\right) = -\frac{1}{a} \frac{\partial p}{\partial \mu},
\]

where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{U}{a} \frac{\partial}{\partial \lambda} + \frac{V}{a} \frac{\partial}{\partial \mu},
\]

and \(p\) is pressure. Cross-differentiate Eqs. (2.1) and (2.2) to obtain the vorticity equation

\[
\frac{D}{Dt} \omega = 0, \quad \omega(\lambda, \mu, t) = 2\Omega \mu + \frac{1}{a} \left\{ \frac{\partial}{\partial \lambda} \left(\frac{V}{1 - \mu^2}\right) - \frac{\partial}{\partial \mu} \left[U(1 - \mu^2)\right] \right\}.
\]

We express the Coriolis term \(f = 2\Omega \mu\) in terms of a vector potential \(A = (a\Omega(1 - \mu^2), 0, 0)\) so that \(f = k \cdot (\nabla \times A)\), where \(k = (0, 0, 1)\). Although a gradient vector of an arbitrary
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A scalar field may be added to $A$ without affecting the result, this particular choice ensures that Kelvin’s circulation around the $\lambda$-periodic contour be defined correctly (Bühler (2009) §10). Then absolute vorticity $\omega$ can be written in terms of the curl of absolute velocity $V^* + A$

$$V^* + A = \left( (U + \Omega a)(1 - \mu^2), \frac{V}{1 - \mu^2}, 0 \right) \equiv (M, N, 0), \quad (2.4)$$

$$\omega = \mathbf{k} \cdot (\nabla \times (V^* + A)) = \frac{1}{a} \left( \frac{\partial}{\partial \lambda} N - \frac{\partial}{\partial \mu} M \right). \quad (2.5)$$

Now Kelvin’s circulation $C$ about a closed material contour can be expressed as

$$C = \oint \left( V^* + A \right) \cdot d\mathbf{l} = \iint_D \omega dS, \quad (2.6)$$

where $d\mathbf{l}$ is the line element vector of the closed material contour and $D$ denotes the domain enclosed by the contour. The last identity is due to the Stokes theorem. Kelvin’s circulation is a conserved quantity for barotropic flow in the absence of forcing and dissipation. Since under that condition absolute vorticity is also material, we consider the circulation about these $\omega$ contours

$$C(\omega) = \iint_{\omega \geq \omega} \hat{\omega} dS. \quad (2.7)$$

The surface integral of vorticity can be estimated easily from a gridded data set by weighted box counting. Simply determine if a given grid point is within the interior of a contour, then add the vorticity at that point times the area of the grid box, which is the spherical quadrangle at that latitude, $\Delta S_j = a^2 \cos \phi_j \Delta \phi \Delta \lambda$:

$$C(\omega) \simeq \sum_{\omega_{ij} \geq \omega} \hat{\omega}_{ij} \Delta S_j, \quad (2.8)$$

where the subscripts $(i, j)$ denote the longitudinal and latitudinal indices for the grid point. On the sphere there is an ambiguity about the definition of interior, so we have chosen the region designated by positive $\omega$ gradient pointing inward. This tends to be a connected region enclosing the north pole, however disconnected islands can occur and their contribution to the circulation should be included. Absolute circulation can be also expressed as a function of equivalent latitude. For a given value of absolute vorticity $\omega = \omega_e$ the corresponding equivalent latitude may be computed from the following relationship:

$$\text{Area}(\omega_e) = \iint_{\omega \geq \omega_e} dS = \int_{\mu_e}^1 2\pi a^2 d\mu = 2\pi a^2 (1 - \mu_e) \simeq \sum_{\omega_{ij} \geq \omega_e} \Delta S_j. \quad (2.9)$$

Note that the last summation may be evaluated easily by box counting.

Absolute vorticity is a monotonic function of equivalent latitude by construction, and this function is invariant with time in the absence of dissipation and forcing. Hence $\omega(\mu_e)$ serves as a time-independent reference state, as does $C(\mu_e)$, which is at the core of the theory developed by NZ10a and NS10ab. In reality (and in a numerical model) $\omega$ is not conserved exactly, but its contours provide a practical quasi-material coordinate which allows one to construct a continuously re-initialized version of the GLM theory (modified Lagrangian mean, McIntyre (1980)).
2.2. Generalized Lagrangian Mean pseudomomentum

Now we introduce the basic notions of GLM theory. For a more complete exposition the reader should refer to Andrews & McIntyre (1978b) and Bühler (2009). We start by constructing a one-to-one mapping between the particle locations on a zonal circle defined by \( x = (\lambda, \mu_m) \) and those on a wavy contour of \( \omega \), expressed as \( x + \xi = (\lambda + \Delta\lambda, \mu_m + \Delta\mu) \) at a given time, where \( \Delta\lambda(\lambda, \mu_m) \) and \( \Delta\mu(\lambda, \mu_m) \) are the departure of the latter relative to the former. The Lagrangian zonal average along the contour and the corresponding eddy quantity for a scalar \( \chi \) are defined as

\[
\overline{\chi}(\mu_m, t) = \frac{1}{2\pi} \int_0^{2\pi} \chi(\lambda + \Delta\lambda, \mu_m + \Delta\mu, t) d\lambda.
\]  

(2.10)

\[
\chi'(\lambda, \mu_m, t) \equiv \chi(\lambda + \Delta\lambda, \mu_m + \Delta\mu, t) - \overline{\chi}(\mu_m, t).
\]  

(2.11)

Notice that the integral in Eq. (2.10) is defined with respect to the particle positions on the zonal circle at \( \mu = \mu_m \), which are uniform in \( \lambda \), but the distribution of the corresponding particles on the wavy contour is not necessarily uniform. Therefore (2.10) differs from an unweighted average along the wavy contour; rather it should be thought of as a density-weighted average over a wavy infinitesimal tube

\[
\overline{\chi} = \int \frac{\chi}{|\nabla_\omega|} dl / \int \frac{dl}{|\nabla_\omega|} \approx \sum_{\omega + \Delta\omega > 2i\omega} \chi_{ij} \Delta S_j / \sum_{\omega + \Delta\omega > 2i\omega} \Delta S_j.
\]  

(2.12)

where the circuit integrals are defined on the \( \omega \)-contour and the last approximation is used to evaluate \( \overline{\chi} \) from gridded data. Now if we choose \( \xi = (\Delta\lambda, \Delta\mu) \) and \( x \) such that \( \Delta\lambda = \Delta\mu = 0 \) so \( \xi \) is an eddy quantity, then by substituting \( \chi = \mu = \mu_m + \Delta\mu \) in Eq. (2.10), one obtains \( \overline{\chi} = \mu_m \). This means that \( \mu_m \) is the density-weighted average, or the center-of-mass, latitude of the wavy material tube. This construction is at the heart of the GLM theory, allowing the material tendency to retain their exact structure upon Lagrangian averaging with no additional eddy terms

\[
\frac{\partial}{\partial t} \mu_m - \frac{\mu_m}{\lambda} \frac{\partial}{\partial \mu_m} \]  

(2.13)

where \( \nabla L = \frac{\mu_m}{\lambda} \frac{\partial}{\partial \mu_m} \) is the Lagrangian-mean velocity with which the center-of-mass latitude moves. Note that \( \mu_m \) is to be distinguished from the equivalent latitude \( \mu_e \), which is the latitude of the \( \omega \)-contour in an eddy-free reference state. The difference between the two reflects, to the lowest order, the latitudinal Stokes drift of the contour. Using Eq. (2.13), Kelvin’s circulation theorem may be written as

\[
\frac{\partial}{\partial t} C + \frac{1}{2\pi d} \bigg[ \frac{\partial}{\partial t} + \frac{\nabla L}{a} \frac{\partial}{\partial \mu_m} \bigg] C = 0,
\]  

(2.14)

where \( C \) is defined in Eq. (2.6) and we write it out using Eq. (2.4) and the expression for the line element vector \( dl = a \frac{\partial}{\partial \lambda} (\lambda + \Delta\lambda, \mu_m + \Delta\mu) d\lambda \) as

\[
\frac{1}{2\pi d} C = \frac{1}{2\pi d} (\nabla^L + A) \cdot (dl/d\lambda)^L = M(\lambda + \Delta\lambda, \mu_m + \Delta\mu) \frac{\partial (\lambda + \Delta\lambda)}{\partial \lambda} + N(\lambda + \Delta\lambda, \mu_m + \Delta\mu) \frac{\partial (\mu_m + \Delta\mu)}{\partial \lambda} = (M^L + M') (1 + \frac{\partial (\Delta\lambda)}{\partial \lambda}) + (N^L + N') \frac{\partial (\Delta\mu)}{\partial \lambda}.
\]  

(2.15)
The sum of the last two terms is defined as \(-P(\mu_m, t) \cos \phi_m\), where \(P\) is the GLM pseudomomentum. Thus

\[
\frac{1}{2\pi a} C = \mathcal{M}^L - P \cos \phi_m, \tag{2.15}
\]

where the sign convention for the pseudomomentum is standard. By applying the circulation theorem \((2.14)\) to \((2.15)\), we have

\[
\frac{\mathcal{D}^L}{Dt} \mathcal{M}^L = \frac{\mathcal{D}^L}{Dt} (P \cos \phi_m). \tag{2.16}
\]

Thus, if pseudomomentum is steady following the center-of-mass latitude, there will be no acceleration of Lagrangian-mean angular momentum. This is the exact GLM nonacceleration theorem. Given Eq. \((2.15)\) and that Eq. \((1.8)\) can be also written as

\[
\frac{1}{2\pi a} C = \mathcal{M} + A^* \cos \phi_v, \tag{2.17}
\]

it is clear that the GLM and new formalisms differ only in the partition of Kelvin’s circulation and that the conservation of circulation leads to the respective forms of nonacceleration theorem \(((1.9)\) and \((2.16)\)).

### 2.3. Small-amplitude theory

To see the relationship between \(P\) and \(A^*\) more explicitly, we will develop a small-amplitude theory. Suppose a material \(\omega\)-contour is disturbed slightly from a zonally symmetric state \((\lambda, \mu_e)\) into a wavy form \((\lambda + \Delta \lambda, \mu_e + \Delta \mu)\). The difference between the Eulerian and Lagrangian means induced by the waviness of the material contour is expressed as a Stokes’ correction

\[
\chi^L = \chi^L - \chi = \frac{\Delta \lambda}{\partial \lambda} \frac{\partial \chi^L}{\partial \lambda} + \Delta \mu \frac{\partial \chi^L}{\partial \mu_m} + \frac{1}{2} (\Delta \mu)^2 \frac{\partial^2 \chi^L}{\partial \mu_m^2} + O(\alpha^3) \tag{2.18}
\]

as \(\alpha \to 0\). This expression is simply a Taylor expansion of the Lagrangian mean through second order in \(\alpha\). Notice that we use \(\partial / \partial \mu_m\) instead of \(\partial / \partial \mu_e\) for the purpose of the subsequent development but this does not affect the order of accuracy since the difference between \(\mu_e\) and \(\mu_m\) is \(O(\alpha^2)\).

Now we use the fact that \(\partial \mathcal{M} / \partial \mu_m = -a \bar{\omega} + O(\alpha^2)\) to write the Stokes’ correction for \(\mathcal{M}\):

\[
M^S = \frac{\Delta \lambda}{\partial \lambda} \frac{\partial \mathcal{M}^L}{\partial \lambda} + \Delta \mu \frac{\partial \mathcal{M}^L}{\partial \mu_m} + \frac{1}{2} (\Delta \mu)^2 \frac{\partial^2 \mathcal{M}^L}{\partial \mu_m^2} + O(\alpha^3) = \Delta \lambda \frac{\partial \mathcal{M}^L}{\partial \lambda} + \Delta \mu \frac{\partial \mathcal{M}^L}{\partial \mu_m} - \frac{1}{2} a (\Delta \mu)^2 \frac{\partial^2 \mathcal{M}^L}{\partial \mu_m^2} + O(\alpha^3). \tag{2.19}
\]

In the small-amplitude limit the the Lagrangian eddy terms may be approximated as

\[
M^L = M' + \frac{\partial \mathcal{M}}{\partial \mu_m} \Delta \mu + O(\alpha^2), \quad \text{and} \quad N^L = N' + O(\alpha^2).
\]

Substituting these expressions into Eq. \((2.15)\) we find that

\[
\frac{1}{2\pi a} C = \mathcal{M}^L - P \cos \phi_m = \mathcal{M} + M^S - P \cos \phi_m = \mathcal{M} + \frac{\Delta \lambda}{\partial \lambda} \frac{\partial \mathcal{M}^L}{\partial \lambda} + \Delta \mu \frac{\partial \mathcal{M}^L}{\partial \mu_m} - \frac{1}{2} a (\Delta \mu)^2 \frac{\partial^2 \mathcal{M}^L}{\partial \mu_m^2} + (M' + \frac{\partial \mathcal{M}}{\partial \mu_m} \Delta \mu) \frac{\partial \Delta \lambda}{\partial \lambda} + N' \frac{\partial \Delta \mu}{\partial \mu_m} + O(\alpha^3)
\]

\[
= \mathcal{M} + \Delta \mu (\frac{\partial \mathcal{M}^L}{\partial \mu_m} - \frac{\partial \mathcal{M}}{\partial \mu_m}) - \frac{1}{2} a (\Delta \mu)^2 \frac{\partial^2 \mathcal{M}^L}{\partial \mu_m^2} + \frac{1}{2} \frac{\partial \mathcal{M}}{\partial \mu_m} \frac{\partial (\Delta \mu)^2}{\partial \mu_m} + O(\alpha^3)
\]

To get from line 2 to line 3 in the above, we have used integration by parts and the
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fact that \( \partial \Delta \lambda / \partial \lambda + \partial \Delta \mu / \partial \mu_m = O(\alpha^2) \). Since in the conservative small-amplitude limit \( \omega' = -(\partial \omega / \partial \mu_m) \Delta \mu + O(\alpha^2) \), the above can be further rendered as

\[
\frac{1}{2\pi a} C = \frac{1}{2\pi a} \mathcal{C} - \frac{1}{2} \frac{\partial \mathcal{M}}{\partial \mu_m} \frac{\partial (\Delta \mu)^2}{\partial \mu_m} + \frac{1}{2} a(\Delta \mu)^2 \frac{\partial \omega}{\partial \mu_m} + O(\alpha^3),
\]

(2.20)

where \( \mathcal{C}(\mu_m, t) = 2\pi a \mathcal{M}(\mu_m, t) \) is the circulation about the zonal circle at \( \mu = \mu_m \). The second term on the right-hand side can be further transformed by noting \( C/2\pi a = \mathcal{M} + O(\alpha^2) \), so Eq. (2.20) becomes

\[
C = \mathcal{C} - \frac{1}{2} \frac{\partial (\Delta \mu)^2}{\partial \mu_m} \frac{\partial C}{\partial \mu_m} + 2\pi a^2 \frac{\partial (\Delta \mu)^2}{\partial \mu_m} \frac{\partial \omega}{\partial \mu_m} + O(\alpha^3).
\]

(2.21)

Now if we take the time derivative assuming that the tendency of mean quantities is \( O(\alpha^2) \) and note that

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \frac{\partial (\Delta \mu)^2}{\partial \mu_m} \frac{\partial C}{\partial \mu_m} \right] = \frac{V^L}{a} + O(\alpha^3),
\]

then we have an expression for the Lagrangian derivative of the circulation

\[
\left[ \frac{\partial}{\partial t} + \frac{V^L}{a} \frac{\partial}{\partial \mu_m} \right] C = \frac{\partial}{\partial t} \mathcal{C} + \frac{\partial}{\partial t} \left( 2\pi a^2 \frac{\Delta \mu^2}{2} \frac{\partial \omega}{\partial \mu_m} \right) + O(\alpha^3).
\]

(2.22)

Note that the Eulerian terms on the right-hand side cancel

\[
\frac{\partial}{\partial t} \mathcal{C} = 2\pi a^2 V^L \omega^f \quad \text{and} \quad \frac{\partial}{\partial t} \left( 2\pi a^2 \frac{\Delta \mu^2}{2} \frac{\partial \omega}{\partial \mu_m} \right) = -2\pi a^3 V^L \omega^f + O(\alpha^3).
\]

Hence we have recovered the small-amplitude approximation to Eq. (2.14)

\[
\left[ \frac{\partial}{\partial t} + \frac{V^L}{a} \frac{\partial}{\partial \mu_m} \right] C = O(\alpha^3)
\]

(2.23)

and verified that Eq. (2.20) is consistent with the GLM circulation theorem.

Now we return to Eq. (2.20) and consider the first two terms as a Taylor expansion of \( C \) about latitude \( \mu_m \), then the circulation may be rewritten as

\[
\frac{1}{2\pi a} C = \frac{1}{2\pi a} \mathcal{C} \left( \mu_m - \frac{1}{2} \frac{\partial (\Delta \mu)^2}{\partial \mu_m} \right) + \frac{1}{2} a(\Delta \mu)^2 \frac{\partial \omega}{\partial \mu_m} + O(\alpha^3).
\]

(2.24)

The second term on the right-hand side differs from \( A^* \cos \phi_e \) only by \( O(\alpha^3) \) because at small amplitude (NS10a appendix)

\[
A^*(\mu_e, t) = \frac{a}{2 \cos \phi_e} \frac{d \omega}{d \mu_e} \frac{(\Delta \mu)^2}{2} + O(\alpha^3).
\]

(2.25)

Thus,

\[
\frac{1}{2\pi a} \left[ C(\mu_m, t) - \mathcal{C} \left( \mu_m - \frac{1}{2} \frac{\partial (\Delta \mu)^2}{\partial \mu_m}, t \right) \right] = A^*(\mu_e, t) \cos \phi_e + O(\alpha^3).
\]

(2.26)

By comparing the above with Eq. (1.8), we see that \( \mathcal{C} \left( \mu_m - \frac{1}{2} \frac{\partial (\Delta \mu)^2}{\partial \mu_m}, t \right) \) is the zonal-mean circulation at the equivalent latitude, which is given by the center-of-mass latitude of the \( \omega \)-contour \( \mu_m \) minus its Stokes correction. Applying similar approximations to Eq.
(2.15), we arrive at another equation involving \(-P\)
\[
\frac{1}{2\pi a} \left[C(\mu_m, t) - C(\mu_m, t)\right] - \frac{\partial (M' \Delta \mu)}{\partial \mu_m} + \frac{1}{2} a(\Delta \mu)^2 \frac{\partial \alpha}{\partial \mu_m} = -P(\mu_m, t) \cos \phi_m + O(\alpha^3),
\]
where the last two terms on the left-hand side are the Stokes correction to \(\overline{M}\). By taking the difference between Eqs. (2.26) and (2.27) we have
\[
-P(\mu_m, t) \cos \phi_m = 2A^*(\mu_e, t) \cos \phi_e - \frac{\partial (M' \Delta \mu)}{\partial \mu_m} - \frac{1}{2} \frac{\partial (\Delta \mu)^2}{\partial \mu_m} \overline{M} + O(\alpha^3).
\]
Thus, apart from the two terms associated with the Stokes correction, \(-P\) approaches twice the wave activity \(A^*\) at small amplitude. (The second \(A^*\) arises from the last term on the left-hand side of Eq. (2.27)). However, in practice the Stokes correction terms (especially the second one) tend to dominate pseudomomentum, causing \(A^*\) and \(-P\) to be very different (for example, \(-P\) is not sign-definite) as we will see in §4.2 (see also Shepherd (1983) and Uryu (1979) for related discussions).

3. Impulse-Casimir methods

As the conservation of Kelvin's circulation leads directly to the nonacceleration theorem involving the GLM pseudomentum, the conservation of Kelvin's impulse has been exploited to construct another class of eddy metrics. The measure of local wave activity introduced by Killworth & McIntyre (1985) is based on the impulse-Casimir method and can be defined as
\[
A_{IC}(\omega_R, \omega_e) \cos \phi = -a \mu_e \omega_e + I(\omega_R + \omega_e) - I(\omega_R),
\]
where \(\omega_R(\mu)\) is a zonally symmetric, time-independent reference state of absolute vorticity and \(\omega_e(\lambda, \mu, t) = \omega(\lambda, \mu, t) - \omega_R(\mu)\) is the departure from that reference state. (Note that generally \(\omega_e \neq \omega^*\) since the former may contain a nonzero zonal-mean component.) McIntyre & Shepherd (1987) show that the above construction is based on the conservation of the density of Kelvin's impulse \(\mu \omega\) and the impulse-Casimir function
\[
I(\omega) = \int_{\omega_{\text{min}}}^{\omega} a \mu_R(\omega) d\omega.
\]
Here \(\omega_{\text{min}}\) is the minimum value of \(\omega_R\) and \(\mu_R(\omega)\) is the inverse of \(\omega_R(\mu)\), where we assume that \(\omega_R\) is a monotonic function of \(\mu\) so that its inverse is uniquely defined. The choice of \(\omega_R\) is arbitrary: as an example, let \(\omega_R = 2\Omega \mu\) (solid-body rotation) as shown by the solid line in figure 1. Then from (3.1) \(A_{IC} \cos \phi\) at \((\lambda, \mu_0, t)\) may be interpreted geometrically as the area in the \((\mu, \omega)\)-plane bounded by the curves \(\omega = \omega_R(\mu)\), \(\omega = \omega_R(\mu_0) + \omega_e\), and \(\mu = \mu_0\) as depicted by area (i) in figure 1. This area is a right triangle with base \(\omega_e\) and height \(a\omega_e/(2\Omega)\) so that
\[
A_{IC} \cos \phi(\lambda, \mu_0, t) = \frac{1}{2} a(\omega_e)^2/(2\Omega).
\]
This provides a two-dimensional distribution of wave activity density on the sphere. In contrast, \(A\) and \(A^*\) do not provide any information about the longitudinal distribution. To facilitate the comparison, we will take the zonal average of \(A_{IC}\). In the small-amplitude limit, the zonal average of (3.3) is identical with (1.3), so \(\overline{A_{IC}}\) converges to both \(A\) and \(A^*\). As shown by Killworth & McIntyre (1985), \(A_{IC}\) satisfies local flux conservation. Its
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Figure 1. The densities of several wave activities are plotted in the \((\mu, \omega)\)-plane. The solid-body-rotation reference state \(\omega_R = 2\Omega\mu\) is plotted as a thin black line. The instantaneous \(\omega(\mu_e, t)\) is the curved dash-line (for this curve the horizontal axis is \(\mu_e\)). The triangular areas enclosed in gray (i and ii) represent the impulse-Casimir wave activity densities at \((\lambda, \mu_0, t)\). The area enclosed in black (iii) is the density of \(A^*\) if wave amplitude is a constant function of latitude. See text for details.

Zonally averaged form is

\[
\frac{\partial}{\partial t} \tilde{A}_{IC} \cos \phi = -\frac{1}{a} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) (\mu \nu_v + \overline{v_e} A_{IC}) \right],
\]

(3.4)

where the last term denotes the eddy advection of wave activity, which corresponds to the right-hand side of Eq. (1.4).

Another second-order wave property may be defined in terms of the impulse-Casimir function of latitude

\[
\tilde{A}_{IC}(\mu_R, \mu_e) \cos \phi = -a \omega_R(\mu) \mu + \tilde{I}(\mu_R + \mu_e) - \tilde{I}(\mu_R),
\]

(3.5)

where \(\mu_e = \mu - \mu_R(\omega)\) and

\[
\tilde{I}(\mu) = \int_{-1}^{\mu} a \omega_R(\tilde{\mu}) d\tilde{\mu}.
\]

(3.6)

Again we may interpret \(\tilde{A}_{IC} \cos \phi\) at \((\lambda, \mu_0, t)\) geometrically as the area in the \((\mu, \omega)\)-plane labeled (ii) in figure 1, which is bounded by \(\omega = \omega_R(\mu), \omega = \omega_0 = \omega_R(\mu_0)\) and \(\mu = \mu_0 + \mu_e\). This triangular area is equal to that of area (i) because the chosen \(\omega_R\) is linear in \(\mu\)

\[
\tilde{A}_{IC} \cos \phi = \frac{1}{2} a^2 \Omega(\mu_e)^2 = \frac{1}{2} a (\omega_e)^2 / (2\Omega) = A_{IC} \cos \phi,
\]

(3.7)

but if \(\omega_R\) is not a linear function of \(\mu\), \(A_{IC} \neq \tilde{A}_{IC}\) in general.

One might attempt to extend the above geometrical argument to \(A^*\) by considering the expression

\[
\tilde{A}^*(\mu_e, \mu_e) \cos \phi_e = -a \omega(\mu_e, t) \mu_e + \tilde{I}^*(\mu_e + \mu_e) - \tilde{I}^*(\mu_e),
\]

(3.8)
where $\mu_e = \mu - \mu_e(\omega)$ is the difference in the local equivalent latitude between the observed state and the eddy-free reference state and

$$I^*(\mu) = \int_{-1}^{\mu} \omega(\mu_e, t) d\mu_e. \quad (3.9)$$

Note that $\omega_R(\mu)$ and $\mu_R$ have been replaced by $\omega(\mu_e, t)$ and $\mu_e$, respectively. They would be identical if they were initially identical and the dynamics were conservative. In that case $A^* \cos \phi_e = A_{IC} \cos \phi$. In reality, even if they are identical initially, $\omega(\mu_e, t)$ will depart from $\omega_R(\mu_e)$ as nonconservative effects accumulate (figure 1). Generally, $A^* \cos \phi_e$ is equal to the area (iii) enclosed by the black curve in figure 1, bounded by $\omega = \omega(\mu_e)$, $\omega = \omega(\mu_0)$, and $\mu = \mu_0 + \mu_e$. However, the real limitation to this geometrical interpretation is that the zonal average of $A^* \cos \phi_e$ departs from $A^* \cos \phi$ when eddies attain finite amplitude. This is because the integrals in $A^*$ is defined in $(\lambda, \mu_e)$-space, whereas the area integral in Eq. (1.7) is defined in $(\lambda, \mu)$-space. In order to connect $A^* \cos \phi_e$ with $A^* \cos \phi$, the terms in (3.8) must be redefined in $(\lambda, \mu)$-space, which requires that integrals with respect to $\mu_e$ be weighted locally by a Jacobian $\partial \mu / \partial \mu_e$. Unfortunately the Jacobian departs from unity as eddy amplitude increases, and it becomes infinite where the $\omega$-contour overturns. As a result, $A^* \cos \phi_e$ in $(\lambda, \mu)$-space diverges quickly from $A_{IC} \cos \phi$, causing the departure of $A^* \cos \phi_e$ from $A_{IC} \cos \phi$. In principle it is possible to have $\partial \mu / \partial \mu_e = 1$ everywhere if the displacement amplitude $\mu_e(\lambda, \mu, t)$ is constant in $\mu$. Of course, not only is this incompatible with the overturning of the $\omega$-contours, but it cannot be realized on the sphere at all, because $\mu_e$ must vanish at the poles. (It can be realized on the doubly periodic $\beta$-plane as a very special case.) The large variation in the Jacobian is the leading cause for the two wave activity densities to diverge at finite amplitude.

On the other hand, in the conservative small-amplitude limit, even if $\omega_R$ is a nonlinear function of $\mu$, the curvature in the reference state is negligible over the scale of displacement, the Jacobian is close to unity, and $\omega(\mu_e, t) = \omega_R(\mu)$. In this case the areas (i), (ii), and (iii) in figure 1 converge. This explains the convergence of the two wave activities in the small-amplitude limit.

4. A numerical experiment

In this section we will conduct an idealized numerical experiment, in which an eddy forcing is applied to drive a zonal jet on the rotating sphere. We will then use the result of the experiment as a controlled test bed for the various forms of wave activity diagnostics introduced earlier. The vorticity equation (2.3) can be written in terms of a streamfunction $\psi$ since $(U, V)$ is nondivergent:

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + \frac{1}{a^2} \left( \frac{\partial \psi}{\partial \lambda} \frac{\partial \omega}{\partial \mu} - \frac{\partial \psi}{\partial \mu} \frac{\partial \omega}{\partial \lambda} \right) = Q, \quad (4.1)$$

where $Q$ represents forcing and dissipation. This equation is discretized with a standard spectral transform method truncated at T85 on a $256 \times 128$ Gaussian grid. The size and the rotation rate of the sphere are chosen to be those of the earth. The time-stepping is done with an Adams-Bashforth third-order scheme and a time step of eight minutes. A 6th-order hyperviscosity is applied to damp the highest order modes with an e-folding timescale of one day (Durran (2010) §6).

To simulate midlatitude stirring by planetary waves, a meridionally localized, stationary vorticity forcing is applied to a barotropic layer initially in solid body rotation. The
forcing is slowly increased then maintained at a constant amplitude for thirty days after which it is slowly reduced and the system is allowed to evolve freely for another thirty days. The forcing has a Gaussian profile in latitude, centered at 45°N with a width of 20 degrees and sinusoidal, wavenumber four amplitude in the longitudinal direction. The maximum amplitude of the forcing is $2.5 \times 10^{-10} \text{s}^{-2}$ which is found to produce a reasonable amount of stirring without generating barotropic instability early in the experiment as to obscure the wave dynamics. The specific form of $Q$ is

$$Q = \gamma_\zeta + \kappa \nabla^6 \omega$$

where

$$\gamma_\zeta(\lambda, \mu, t) = \begin{cases} 
\gamma_0(t) & 35^\circ \leq \phi \leq 55^\circ \\
0 & |\phi - 45^\circ| > 10^\circ,
\end{cases}$$

where the shape of $\gamma_0$ is sketched in figure 4b below.

4.1. Mean flow, wave activity, and reference state

The formation of a jet is illustrated in figure 2a with profiles of the zonal mean wind averaged over three twenty-day intervals. Westerlies localized in the region of forcing develop progressively throughout the the simulation with easterlies on both flanks. The asymmetry in the easterlies is pronounced with broader and stronger westward flow on the poleward side of the jet. As the jet emerges, finite-amplitude cyclone/anti-cyclone
pairs propagate away from the forcing region. The associated perturbation to the initial distribution of planetary vorticity is seen in figure 2b. Within the first twenty days a large positive gradient in $\varpi$ develops at the axis of the jet, with reversals in the gradient in the regions of westward flow (figures 2b,d). These negative gradients lead to barotropic instability which produces strong mixing on the flanks of the jet. Figure 2c indicates large convergence of momentum into the axis of the jet early in the simulation with a symmetric pattern of divergence on the flanks. Later in the experiment this symmetry is broken, giving way to a more complex pattern of eddy forcing. This results in the evolution of the zonal-mean wind beyond the termination of the forcing and even continuing at the end of the integration.

To sort out the effects of advective eddy-mean flow interaction and those of forcing and mixing, we show in figure 3 $u_{ref}(\phi_e)$ and $\omega(\phi_e)$, to be compared with $\overline{\varpi}(\phi)$ and $\overline{\varpi}(\phi)$ in figure 2. As noted in §1 and by NZ10ab, these quantities evolve only in response to forcing and mixing, and the difference with the zonal-mean quantities largely reflects the advective eddy-mean flow interaction. In particular, NZ10ab show that the tendency of $u_{ref}$ due to mixing is always negative, as the diffusive flux of vorticity acts as a damping on $A^\ast$.

It is seen that $u_{ref}$ (figure 3a) is greater in magnitude than $\overline{\varpi}$ in figure 2a throughout the domain during each interval. This reflects $u_{ref} = \overline{\varpi} + A^\ast$ and $A^\ast \geq 0$. $u_{ref}$ attains its maximum of more than 8.5 m/s before day 20 and remains fairly constant near the jet axis after about day 30. The gain in $u_{ref}$ is due to forcing, and it extends beyond the region of forcing into high equivalent latitudes (dotted and dashed curves in figure 3a). This is because the waviness in absolute vorticity allows high-equivalent-latitude fluid to migrate into the forcing region. The main evolution of $u_{ref}$ after initial growth is a progressive narrowing of the jet as $A^\ast$ is dissipated on the flanks, reflected in the decrease in $u_{ref}$ from the dashed- to solid curve, with greater dissipation occurring on the poleward side. The profile of absolute vorticity in figure 3b remains (by construction) a monotonic function of equivalent latitude, but it shows an irreversible change in gradients as a result of forcing and mixing. In particular, the enhanced mixing at the flanks of the jet markedly reduce the gradients there, reinforcing the strong gradient at the jet axis and producing a staircase-like profile (Dritschel & McIntyre (2008), NZ10b).

It is not common to see wave activity plotted as a diagnostic of the fluid motion. The conventional analysis of a changing mean flow involves calculating the convergence of eddy momentum flux, or equivalently the divergence of Eliassen-Palm (E-P) flux, arising

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**Figure 3.** (a) $u_{ref}(\phi_e)$ in the Northern Hemisphere averaged over the same three, twenty-day intervals as in figure 2. (b) $\omega(\phi_e)$, the distribution of absolute vorticity with equivalent latitude, also averaged over the same twenty day intervals.
Figure 4. (a) Zonal mean zonal wind (solid line) and E-P flux divergence (dashed line), averaged within the forcing region (35°N-55°N) as a function of time in days. (b) $u_{ref}$ (solid line) and wave activity density (dashed line) also averaged over the forcing region. The dotted line shows the timing of forcing, with a maximum amplitude of $2.5 \times 10^{-10} \text{s}^{-2}$ and a minimum of zero.

from Eqs. (1.5) and (1.6)

$$
\frac{\partial \pi}{\partial t} = -\frac{1}{a \cos \phi} \frac{\partial}{\partial \mu} \left( u'v'(1 - \mu^2) \right). \quad (4.2)
$$

The E-P flux divergence determines the tendency of both $\pi$ and $A^*$ (Eq. (1.5)), however without wave activity their relative magnitude is unknown. In figure 4 we compare this standard flux-based analysis of wave forcing with a direct assessment of wave activity, averaged within the forcing region (35°N-55°N). As seen in figure 4a, the E-P flux divergence (dashed line) is positive for the first 15 days and the zonal-mean zonal wind (solid line) increases within the forcing region. After that point barotropic instability ensues due to the vanishing of the absolute vorticity gradient and the E-P flux divergence begins to oscillate, whilst $\pi$ slowly continues to grow for the duration of the simulation.

In figure 4b the evolution of the reference state flow $u_{ref}$ (solid line) and wave activity $A^*$ (dashed line) within the forcing region are plotted. Since $u_{ref} = \pi + A^*$, these curves represent a partitioning of $u_{ref}$ into eddy and zonal-mean component. Initially $u_{ref}$ grows rapidly and this is almost entirely due to the growth of wave activity through forcing. The two curves only start to diverge around day 5, when $\pi$ becomes significant. After about day 8 wave activity equilibrates and starts to oscillate until the end of the forcing period around day 35. The initial equilibration is due to the balance between the gain through forcing and the loss through E-P flux convergence, but the subsequent oscillation is also affected by dissipation due to mixing. The reference-state flow $u_{ref}$ equilibrates much later ($\approx$ day 20) as the forcing is balanced by the dissipation. After the forcing has been turned off wave activity steadily declines until the end of the simulation. A significant fraction of this decline is due to conversion to $\overline{\pi}$ through the E-P flux convergence, evident in the increase in $\overline{\pi}$ in figure 4a. In comparison, $u_{ref}$ decreases much more slowly than $A^*$. The decrease in $u_{ref}$ in this freely evolving stage primarily reflects the dissipation of wave activity due to mixing. The increasing $\overline{\pi}$ and decreasing $A^*$ boosts the fraction of the former in $u_{ref}$ toward the end of the simulation, making the flow less wavy, although $A^*$ is greater than $\overline{\pi}$ until the forcing is turned off. Such assessment of the eddy-mean partitioning of momentum is not accessible through the E-P flux analysis alone.
Figure 5. (a) Global-mean wave activities (in m/s) for the jet simulation as a function of time: linear wave activity $A$ (dot-dashed), finite-amplitude impulse-Casimir wave activity $A_{IC}$ (dashed) and $A^*$ (solid). Dotted line represents $A$ calculated with instantaneous gradient of zonal-mean absolute vorticity. (b) Time-averaged (days 0-67) wave activities as functions of latitude using the same line markers as in (a). Note that the coordinate for $A^*$ is equivalent latitude.

4.2. Wave activity comparison

We have seen that $A^*$ can provide additional insight for interpreting an evolving flow. Now we compare it with other forms of wave activity using the same result of the above simulation. In figure 5a the time dependence of the global-mean $A^*$ is shown (solid curve), together with finite-amplitude impulse-Casimir wave activity $A_{IC}$ (dashed) and the wave activity of the linearized problem, $A$ (dot-dashed). We use Eq. (1.3) to define $A$, by taking the initial condition (solid body rotation with no wind) as the reference state $d\omega_R/d\mu = 2\Omega$, and remove the zonal-mean component from $\zeta'$. We also use the same reference state for $A_{IC}$, and hence its expression given by (3.7). The three curves overlap until about day 8, which confirms their convergence in the small-amplitude limit. After that they begin to diverge from one another. The linear wave activity $A$ remains smaller in amplitude than the others, while the two finite-amplitude measures depict wave growth until day 20 when barotropic instability begins to alter the flow. At day 20, the global-mean $A_{IC}$ and $A$ are roughly 80 and 50 percent of $A^*$, respectively. The slight upturn after day 30 is associated with the ramp-down of the forcing, which induces some transient effects that linger until around day 40. During the final twenty days $A_{IC}$ and $A^*$ show excellent agreement and remain fairly constant as would be expected in the absence of forcing.

Comparing (1.3) and (3.7) it is clear that the difference between $A$ and $A_{IC}$ is due to the contribution from $\zeta^2/2\Omega$, namely the part of $A_{IC}$ associated with the zonal-mean component of $\omega$, that is spun up by the eddy-mean flow interaction. This is also related to the fact that the gradient of planetary vorticity is greater than the instantaneous $\partial \zeta/\partial \mu$ in the stirring regions where wave amplitude is large. Therefore as a comparison, we show the wave activity linearized about the instantaneous gradient of zonal-mean absolute vorticity (i.e., replace $d\omega_R/d\mu$ in (1.3) by $\partial \zeta/\partial \mu$) in the dotted curve. This linear diagnostic reproduces $A^*$ remarkably well until about day 14, much longer than $A$ or even $A_{IC}$, suggesting the importance of flow profiles for computing wave activity accurately. However, this measure of wave activity lacks conservation properties of $A$ or $A_{IC}$ and breaks down as soon as $\partial \zeta/\partial \mu$ vanishes due to stirring by eddies, which causes the wild oscillation after day 14, rendering it nearly useless.
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Figure 6. Time-latitude diagrams for three wave activity measures. (a) Wave activity $A^*$ as a function of time and equivalent latitude. (b) The zonally averaged $A_{JC}$ as a function of time and latitude. (c) Same as (b) but for $A$. All panels are plotted with the same grayscale, with a contour interval of 1 m/s.

In figure 5b the time averaged distribution of wave activity with latitude shows several important differences between these three diagnostics. The linear measure $A$ indicates a maximum in wave activity at the center of the forcing region, while the two finite-amplitude measures exhibit local minima there. The intriguing distinction between $A_{JC}$ and $A^*$ is that one indicates a more focused, symmetric distribution with respect to the jet, while the other shows a pronounced maximum on the poleward side of the forcing region.

The comparison is further substantiated by figure 6, in which the meridional profiles of $A^*$, $A_{JC}$ and $A$ are plotted as functions of time. Each measure shows the tendency of wave activity to split into a subtropical and subpolar tracks, similar to the barotropic decay of pseudomomentum observed in Held & Phillips (1987). Only $A^*$ shows a clear separation of the two tracks with a persistent minimum in between, where the flow is accelerated. Also $A^*$ exhibits greater amplitude and broader extent in the poleward branch, consistent with the enhanced deceleration observed in that region (figure 6a). In contrast, $A_{JC}$ shows two narrow tracks on the poleward side of the jet, the main track near the axis and a weaker track in higher latitudes (figure 6b). This secondary track is also captured by $A$ but the strong signals on the immediate flanks of the jet are not, implying that they are largely due to mean flow modification (figure 6b,c). Because $A$ lacks these signals on both sides of the jet axis around days 15-30, its time-mean profile in figure 5b shows maximum, not minimum, at the jet axis. Thus, despite the similar behaviors in the global mean, $A^*$ and $A_{JC}$ show substantial differences in their meridional profiles. The differences are largely due to the nonlocal nature of equivalent
Figure 7. (a) Equivalent latitude $\mu_e$ for the Northern Hemisphere as a function of time and absolute vorticity. Contour interval is 0.025. (b) Same as (a) but for center-of-mass latitude $\mu_m$. (c) Same as (b) but for $\mu_m - \mu_e$. Negative values are plotted in dashed contours. (d) Same as (a) but for $A^*$. Contour interval is 0.5 m/s. (e) Same as (d) but for the negative of GLM pseudomomentum $-P$. Contour interval is 15 m/s and negative values are dashed. (f) Same as (d) but for $-P - 2A^*$. Contour interval is 15 m/s.

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latitude, which absorbs the flux of wave activity in Eq. (3.4). $A^*$ is more closely related to the mean flow modification shown in figure 2a than $A_{IC}$.

In figure 7 we also compare $A^*$ with the GLM pseudomomentum $P$. First, we plot equivalent latitude $\mu_e$ (figure 7a) and the center-of-mass latitude $\mu_m$ (figure 7b) for $\omega$-contours in the Northern Hemisphere as a function of time and the value of $\omega$. The center-of-mass latitude is computed using Eq. (2.12) with $\chi_{ij} = \mu_j$.

Both $\mu_e$ and $\mu_m$ are initially linear with $\omega$, but the contours of $\mu_e$ quickly spreads in the forcing region. This is consistent with the increase in $\partial \omega / \partial \mu_e$ in the same region (figure 3b), due to forcing and mixing (without these nonconservative effects, $\mu_e(\omega)$ would be time-independent). Yet by construction $\mu_e$ remains a monotonic function of $\omega$. In contrast, the rearrangement of $\mu_m$ is more complicated: the divergence and convergence of its contours do not necessarily coincide with those of the $\mu_e$-contours, and there are some small ‘islands,’ implying that $\omega(\mu_m)$ can be locally multi-valued. This is a well-known difficulty in the GLM analysis, and for this reason we will not use $\mu_m$ as the coordinate
Wave activity $A^*$ shown in figure 7d essentially duplicates figure 6a using $\omega$ in the ordinate. The corresponding GLM pseudomomentum $-P$ shown in figure 7e in the same coordinate has a very different structure and magnitude. It forms two pairs of negative and positive zones north and south of the forcing region. The pair appearing north to the forcing region is particularly strong, with the maximum value reaching near 120 m/s, an order of magnitude greater than the maximum $A^*$ (note that the contour interval in figure 7e is 30 times greater than that in figure 7d). Compared to $A^*$, the zone of maximum positive values is much narrower and located much lower in latitude. Even in the early stage of simulation, $-P$ is qualitatively different from $A^*$, for example it is predominantly negative in high latitudes. Figure 7f shows $-P - 2A^*$. According to Eq. (2.28), this quantity reflects (in the small-amplitude limit) the effects of the Stokes drift. There is little qualitative difference between figures 7e and f, thus the GLM pseudomomentum is dominated by the Stokes impulse and has little bearing on the wave activity. The greatest contribution to $-P$ is from the second correction term in Eq. (2.28) (impulse associated with the departure of the center-of-mass latitude from the equivalent latitude). This can be seen in $\mu_m - \mu_e$ in figure 7c, whose structure qualitatively matches that of $-P - 2A^*$. In fact, we find that $\mu_m - \mu_e$ times $\partial \overline{M} / \partial \mu_e \approx 2\Omega a \tan \phi_e$ gives a reasonable estimate of $-P - 2A^*$.

4.3. Sensitivity to spectral truncation

To test the sensitivity of the dynamics and diagnostics to the number of retained harmonics, we have repeated the simulation with T63 and T42 truncations. The analyses for these runs are performed using the same 256 × 128 Gaussian grid. The model parameters are identical except for the hyperdiffusion coefficient, which is adjusted to keep the damping rate for the harmonic with maximum resolved wavenumber unchanged.

Figure 8a shows that $u_{ref}$ in the forcing region is relatively insensitive to the truncation (indicated by the three curves near the top with different line types). There is about 10 percent decrease in the peak value from T85 to T42; this is probably due to the enhanced hyperdiffusion in lower truncation runs since $u_{ref}$ responds to only the nonconservative forcing and dissipation. The zonal-mean wind, which plots lower in figure 8a, is nearly identical for T85 and T63 but shows some increase in T42. This suggests that the E-P flux
divergence (eddy vorticity flux) increases at low resolution, which in turn suggests that the small-scale contributions to the vorticity flux filtered from T42 is likely negative and not fully compensated by the enhanced diffusivity. This last point is consistent with the notion that vorticity flux at small scales is diffusion-like (down-gradient) and therefore negative.

The global-mean wave activities in figure 8b first grow due to forcing and then equilibrate due to enhanced dissipation (the effect of E-P flux divergence integrates to zero in the global mean). Both $A^*$ and $A_{IC}$ decrease appreciably from T85 to T42, but the former decreases more than the latter. This is because the dissipation of $A^*$ is attributed to the diffusive flux of PV (NZ10a), which is largely controlled by the effective diffusivity in the barrier region (i.e., the strong vorticity gradient at $\approx 45^\circ$N in figure 3b) (Nakamura (2008)). The greater numerical diffusion in the T42 truncation makes the barrier more diffusive and hence enhances the dissipation of $A^*$. In contrast, the reference state for $A_{IC}$ is time independent and thus no barrier structure is involved, and hence the diffusive PV flux is less sensitive to the truncation.

5. Summary

In recent papers Nakamura and collaborators have introduced a theory for finite-amplitude eddy-mean flow interaction based on a new wave activity, $A^*$, calculated in terms of the meridional displacement of the contours of PV (or absolute vorticity in the case of barotropic flow) relative to the zonal circle of equivalent latitude. In this article we have investigated the connection between the new theory and the extant formalisms, both theoretically and numerically. The main results are the following.

(1) The new wave activity is a close relative to the GLM pseudomomentum (Andrews & McIntyre (1978b)), differing by the impulse associated with the Stokes corrections to the center-of-mass latitude of the PV contour and to the zonal-mean angular momentum in the small-amplitude conservative limit. However, at finite amplitude the Stokes impulse dominates the pseudomomentum and it has little bearing on the wave activity. The use of equivalent latitude (an eddy-free reference state) keeps the Stokes drift from affecting wave activity and achieves a more useful partition of Kelvin’s circulation ($2\pi a \cos \phi_e (\bar{u} + A^* + \Omega a \cos \phi_e)$) than the one offered by the GLM theory.

(2) The new wave activity converges to the impulse-Casimir wave activity $A_{IC}$ (Killworth & McIntyre (1985), McIntyre & Shepherd (1987), Haynes (1988)) and the linear wave activity $A$ in the small-amplitude, conservative limit. At finite amplitude, although the global-mean $A_{IC}$ and $A^*$ behave similarly, their local profiles differ substantially. A geometrical interpretation of $A_{IC}$ illustrates why $A_{IC}$ departs from $A^*$ at finite amplitude: the local mapping from equivalent latitude to geographical latitude becomes nonuniform. Unlike $A_{IC}$, $A^*$ cannot be defined as a local wave activity density, but it satisfies an exact nonacceleration theorem for the zonal-mean flow in the conservative (but finite-amplitude) limit. Thus, $A^*$ has a more direct bearing on the local modification of the zonal-mean flow than $A_{IC}$.

(3) An eddy-free reference state $u_{ref} = \pi + A^*$ is unaffected by the advective flux of eddy vorticity and thus useful for the assessment of nonconservative effects on the flow.

In summary, the theory for finite-amplitude eddy-mean flow interaction based on $A^*$ is a practical alternative to the GLM formalism, which has long been considered unsuitable for data analysis (Shepherd 1983, McIntyre (1980), Andrews et al. (1987)). The application of the new theory has already proven fruitful, allowing one to quantify from meteorological data alone the adiabatic adjustment to the mean flow by the eddies (NS10ab). The barotropic formulation based on Kelvin’s circulation extends naturally to the diagnosis
of atmospheric flows in the isentropic coordinate (NS10b) but it can be easily adapted to baroclinic quasi-geostrophic flows in the isobaric coordinate as well (NS10a). Furthermore, $A^*$ can incorporate the effects of mixing through effective diffusivity (Nakamura (1996)) of PV, thus providing a separate measure of irreversible, nonadiabatic adjustment to the mean flow (NZ10ab).

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