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VARIATIONAL ANALYSIS OF ENERGY-ENSTROPHY THEORIES ON THE SPHERE

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Abstract. Kraichnan's energy-enstrophy theory for 2D inviscid flows on the sphere is discussed within a variational framework. We will give necessary and sufficient conditions for the existence and uniqueness for the extremals of the energy with zero circulation under different values of the temperature parameter β. The unboundedness of the augmented energy functional in this model when β is located in the certain intervals will be shown and related to energy catastrophe of the energy-enstrophy model.

1. **Introduction.** The energy-enstrophy theory of 2D ideal fluid flow is based on (i) finite dimensional approximations of the kinetic energy $H[q]$ and enstrophy $\Gamma[q]$ functionals in terms of truncated Fourier expansions for the vorticity $q \in L^2(S^2)$, (ii) the Gibbs canonical ensemble constructed using these approximations, namely

$$P_s = Z_N^{-1} e^{-\beta E_s}, \tag{1}$$

where the partition function

$$Z_N = \sum_s^N e^{-\beta E_s}$$

is summed over all microstates s in the associated finite-dimensional phase space, and finally (iii) the limit of Z_N as N tends to ∞. Moreover,

$$E_s = E[q_s] = H[q_s] + \frac{\mu}{\beta}\Gamma[q_s] \to E[q]$$

when the number of modes in the truncated Fourier expansions tends to infinity. A nonextensive or continuum limit is known to be valid under this procedure whereby the energy-enstrophy functional

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$$\begin{aligned} E[q] &= H[q] + \frac{\mu}{\beta}\Gamma[q] \\ H[q] &= -\frac{1}{2}\int_{S^2} d\vec{x} \int_{S^2} d\vec{x'} q(\vec{x'})q(\vec{x}) \ln|1 - \vec{x}\cdot\vec{x'}| \\ \Gamma[q] &= \int_{S^2} |q(\vec{x})|^2 d\vec{x} \end{aligned}$$

is recovered when q_s tends to q.

Kraichnan[11] formulated the original energy-enstrophy theory for 2D flows on the plane and used it to discuss the statistics of 2D vortex dynamics. At certain temperatures, the energy-enstrophy functional $E[q]$ of this theory is unbounded. It appears to the authors that any energy-enstrophy theories in which the enstrophy constraints are imposed in a canonical way, will suffer from this difficulty.

More recently, Lim[10] introduced an energy-enstrophy theory where the enstrophy constraint is microcanonical instead of the classical canonical constraint on enstrophy. The non-Gaussian form of the spherical model was chosen to avoid the well-known disadvantages of the Gaussian model in statistical physics.

One of the aims of this paper is to show rigorously in the context of vortex dynamics on the surface of a sphere, the precise way in which the classical energy-enstrophy theory breaks down. To achieve this aim, we study the extremals of the augmented energy within the framework of a continuously relaxed variational problem, and characterize the bifurcations that occurs as the parameter $\frac{\mu}{\beta}$ changes. The approach we take is based on the following general argument:

The thermodynamics of a canonical system that supports negative temperatures converges to the dynamics/statics of the underlying dynamical system at low positive temperatures; in this limit the Helmholtz free energy is dominated by the energy term with the bounded entropy term playing a secondary role.

Although $E[q]$ is only one term in the free energy of the problem, we will show that it plays a role in its low temperature statistical mechanics. Let

$$f[q] = \beta E[q] - S[q] \tag{2}$$

where the entropy functional is given by

$$S[q] = -\int_{S^2} dz\ q(z)\log q(z). \tag{3}$$

Dividing $f[q]$ by β gives the familiar expression for the Helmholtz free energy per particle,

$$\begin{aligned} F[q] &= \beta^{-1} f[q] = \big(E[q] - \beta^{-1}S[q]\big) \\ &= U - TS \end{aligned} \tag{4}$$

if we take the kinetic and potential energies $E[q]$ of inviscid 2D flows on the surface of a sphere to be the internal energy U and $\beta^{-1} = T$ to be the temperature of the statistical mechanics system [12].

Under the low temperature limit,

$$\beta = k\mu \to \infty \text{ with fixed } k > 0,$$

the free energy of the vortex gas converges to the energy:

$$\lim_{\beta = k\mu \to \infty} F[q] = E[q]. \tag{5}$$

The method requires formulating a suitable variational problem for the augmented energy functional $E[q]$ in the Hilbert space $L^2(S^2)$. The variational problem we are concerned with is given by the equations below:

$$E[q] = H[q] + \frac{\mu}{\beta}\Gamma[q], \tag{6}$$

where $E[q]$ is the augmented energy functional. The Hamiltionian functional:

$$H[q] = -\frac{1}{2}\int_{S^2} d\vec{x}\int_{S^2} d\vec{x'}q(\vec{x'})q(\vec{x})\ln|1-\vec{x}\cdot\vec{x'}| \tag{7}$$

is the kinetic energy of the ideal fluid flow corresponding to vorticity $q(\vec{x})$. The enstrophy is the L^2 norm

$$\Gamma[q] = \int_{S^2} |q(\vec{x})|^2 d\vec{x}. \tag{8}$$

The total circulation is fixed at zero, i.e.,

$$\int_{S^2} q(\vec{x})d\vec{x} = 0. \tag{9}$$

In this paper, we will discuss the extremal of $E[q]$ with zero total circulation in the following regimes: (1) $\frac{\mu}{\beta} > 0$; (2) $\frac{\mu}{\beta} < -\frac{||G||}{2}$; (3) $\frac{\mu}{\beta} = -\frac{||G||}{2}$; (4) $-\frac{||G||}{2} < \frac{\mu}{\beta} < 0$, where G is the inverse operator of the Laplacian $-\nabla^2$ on the sphere with zero circulation.

2. Properties of Laplace-Beltrami Operator on the Sphere S^2.

In this section, we present some results of the Laplace-Beltrami operator $-\nabla^2$ on the sphere and find its inverse operator with zero circulation.

2.1. Complete Set of Eigenfunctions.

The spherical harmonics: $Y_{lm}(\theta,\phi)$ satisfy the following important orthonormal property: $\int_{S^2} Y^*_{l'm'}Y_{lm}dS^2 = \delta_{ll'}\delta_{mm'}$. There can only be one complete set of orthonormal functions defined on the unit sphere.

2.2. The Inverse Operator of $-\nabla^2$ on the Unit Sphere S^2.

The stream function satisfies the Poisson equation on the sphere[9]:

$$-\nabla^2\Psi(\vec{x}) = q(\vec{x}). \tag{10}$$

Let G' be the inverse of $-\nabla^2$, i.e., $\Psi(\vec{x}) = (G'q)(\vec{x}) = (Gq)(\vec{x}) + KI_{S^2}(\vec{x})$, where

$$(Gq)(x) = \int_{S^2} \ln\frac{1}{|1-\vec{x}\cdot\vec{x'}|}q(\vec{x'})d\vec{x'}. \tag{11}$$

K is an arbitrary constant of integration and $I_{S^2}(\vec{x})$ is the characteristic function on S^2. Here we set $K = \int_{S^2} q(\vec{x})d\vec{x}$. Thus, we deduce that for the zero total circulation case

$$G^{'} = G. \tag{12}$$

2.3. Properties of the Operator G.

In this section, we list some properties of the operator G.

2.3.1. *G is Compact and Self-adjoint.* Since the kernel of G which is $\ln\frac{1}{|1-\vec{x}\cdot\vec{x'}|}$[13] satisfies $\int_{S^2} d\vec{x}\int_{S^2} d\vec{x'}\left(\ln\frac{1}{|1-\vec{x}\cdot\vec{x'}|}\right)^2 < \infty$, we conclude that the operator G is compact (cf. Griffel[2] page 233 and page 131). The fact that the kernel $\ln\frac{1}{|1-\vec{x}\cdot\vec{x'}|}$ is symmetric implies G is self-adjoint (cf. Griffel page 223).

2.3.2. *Eigenvalues and Eigenfunctions of G.* It is easy to show that: $\int_{S^2} Y_{l\neq 0,m} = 0$ and $\int_{S^2} Y_{00} = \sqrt{4\pi} > 0$. Therefore, for the case of zero circulation, all the $\{Y_{lm}\}$ except Y_{00} are the eigenfunctions of $-\nabla^2$. Since the operator G is the inverse of $-\nabla^2$ on the sphere with zero circulation and the $-\nabla^2$ is positive-definite[2], G and $-\nabla^2$ have the same eigenfunctions and the corresponding eigenvalue of G is the reciprocal of the eigenvalue of $-\nabla^2$ (cf. Griffel[2], page 255) and the eigenvalues of G are all positive.

Remark 2:
The result that $Y_{00} = \sqrt{\frac{1}{4\pi}}$ is also an eigenfunction of G follows from $(GY_{00})(\vec{x}) = \int_{S^2} \ln \frac{1}{|1-\vec{x}\cdot\vec{x'}|} Y_{00} d\vec{x'} = Y_{00} \int_{S^2} \ln \frac{1}{|1-\vec{x}\cdot\vec{x'}|} d\vec{x'}$. Its eigenvalue λ_{00} satisfies: $\lambda_{00} = \int_{S^2} \ln \frac{1}{|1-\vec{x}\cdot\vec{x'}|} d\vec{x'} = 2\pi(2 - 2\ln 2) > 0$. Therefore, all the $\{Y_{lm}\}$ are the eigenfunctions of G and their corresponding eigenvalues are all positive.

3. **The Extremal of $E[q]$ with Zero Circulation when $\frac{\mu}{\beta} > 0$.** In this section, we determine the extremal of $E[q]$ with zero circulation under the condition $\frac{\mu}{\beta} > 0$. Let V be a reflexive Banach Space and $V^{'}$ a nonempty closed and bounded convex subset of V. Let F_1 be a functional which is bounded below and defined on $V^{'}$. It can be shown that:

Theorem 3.1. *If F_1 is strictly convex in V, a necessary and sufficient condition that it has unique minimizer is that its Gâteaux derivative equals to 0.*

Proof. See cf. Eckeland and Temam [3], Prop 1.2 on page 35 □

The set of minimizers of E in $L^2(S^2)$, $minE$, is a closed convex set which is nonempty by assumption. According to *Theorem 3.1*, we will prove the uniqueness and existence of the minimizer of E in the following way:
(1) We will prove E is Gâteaux differentiable and find the Gâteaux derivative.
(2)$E[q] > 0$ for $q \neq 0 \in L^2(S^2)$ when $\frac{\mu}{\beta} > 0$.
(3) We will prove $E[q]$ is strictly convex. The strict convexity of E on $minE$ implies the uniqueness of the minimizer if it exists.
(4) Since E is strictly convex in $L^2(S^2)$, stationary points of E must be unique minimizers of E. We will find the Euler equation of E to get the stationary points, which is equivalent to the vanishing of the Gâteaux.
(5) We observe that the only solution for the Euler equation here is $q \equiv 0$, which implies $q \equiv 0$ is the only minimizer of E when $\frac{\mu}{\beta} > 0$ and $\int_{S^2} q(\vec{x}) d\vec{x} = 0$.

3.1. $E[q]$ **is Gâteaux Differentiable.** Note that

$$\lim_{\lambda\to 0+} \frac{E[q+\lambda p] - E[q]}{\lambda}$$
$$= \int_{S^2} d\vec{x} \int_{S^2} d\vec{x'} p(\vec{x}) q(\vec{x'}) \ln \frac{1}{|1-\vec{x}\cdot\vec{x'}|} + 2\frac{\mu}{\beta} \int_{S^2} p(\vec{x}) q(\vec{x}) d\vec{x}$$
$$= \left\langle p(\vec{x}), \int_{S^2} q(\vec{x'}) \ln \frac{1}{|1-\vec{x}\cdot\vec{x'}|} d\vec{x'} + 2\frac{\mu}{\beta} q(\vec{x}) \right\rangle \tag{13}$$

Here $E'[q] \equiv \int_{S^2} q(\vec{x'}) \ln \frac{1}{|1-\vec{x}\cdot\vec{x'}|} d\vec{x'} + 2\frac{\mu}{\beta} q(\vec{x})$ is a linear functional. Therefore, $E[q]$ is Gâteaux differentiable for any $\frac{\mu}{\beta}$. $E'[q]$ is known as *Gâteaux derivative.*

3.2. $E[q]$ is Bounded Below.

Lemma 3.2. *$H[q] \geq 0$ for all $q \in L^2(S^2)$.*

Proof. We know that

$$\begin{aligned} H[q] &= \frac{1}{2}\langle q, Gq\rangle \\ &= \frac{1}{2}\Big\langle \sum_{l',m'} q_{l'm'}Y_{l'm'}\ , G(\sum_{l,m} q_{lm}Y_{lm})\Big\rangle. \end{aligned} \tag{14}$$

where $q_{lm} = \langle q(\vec{x}), Y_{lm}\rangle$. The expression (14) is bilinear and G is linear, thus we conclude that:

$$\begin{aligned} H[q] &= \frac{1}{2}\Big\langle \sum_{l',m'} q_{l'm'}Y_{l'm'}\ , \sum_{l,m} q_{lm}\lambda_{lm}Y_{lm}\Big\rangle \\ &= \frac{1}{2}\sum_{lm} q_{lm}^2\lambda_{lm}, \end{aligned} \tag{15}$$

where λ_{lm} is the eigenvalue of G corresponding to the eigenfunction Y_{lm}.
Therefore, $H[q] \geq 0$ follows the fact that all the eigenvalues $\lambda_{lm} > 0$. □

Theorem 3.3. *E is positive definite on $L^2(S^2)$ when $\frac{\mu}{\beta} > 0$*

Proof. (i) Since $H[q] \geq 0$, therefore $E[q] = H[q] + \frac{\mu}{\beta}||q(\vec{x})||^2 \geq 0$ for all $q \in L^2(S^2)$ when $\frac{\mu}{\beta} > 0$. (ii) $q \neq 0$ and $\frac{\mu}{\beta} > 0$ implies $\frac{\mu}{\beta}||q(\vec{x})|| > 0$. $H[q] = \frac{1}{2}\sum_{l,m} q_{lm}^2\lambda_{lm} > 0$ follows from the fact that $q_{lm}^2 > 0$ and $\lambda_{lm} > 0$. Therefore, $E[q] = H[q] + \frac{\mu}{\beta}||q(\vec{x})||^2 > 0$ for $q \neq 0 \in L^2(S^2)$. □

Thus, we have proved $E[q]$ is bounded below by 0.

3.3. $E[q]$ is Strictly Convex.

Theorem 3.4. *The energy functional $E[q]$ is strictly convex in $L^2(S^2)$ when $\frac{\mu}{\beta} > 0$.*

Proof. According to the definition of strict convexity, we need to check that

$$E[\lambda p + (1-\lambda)q] < \lambda E[p] + (1-\lambda)E[q] \tag{16}$$

for all $p \neq q \in L^2(S^2)$ and $\lambda \in [0,1]$
By calculation, we know that :

$$E[\lambda p + (1-\lambda)q] = \lambda^2 E[p] + (1-\lambda)^2 E[q] + 2\lambda(1-\lambda)G[p,q],$$

where

$$G[p,q] \equiv \frac{1}{2}\int_{S^2} d\vec{x}\int_{S^2} d\vec{x'}q(\vec{x'})p(\vec{x})\ln\frac{1}{|1-\vec{x}\cdot\vec{x'}|} + \frac{\mu}{\beta}\int_{S^2} q(\vec{x})p(\vec{x})d\vec{x}. \tag{17}$$

We only need to check: $\lambda(1-\lambda)E[p] + \lambda(1-\lambda)E[q] - 2\lambda(1-\lambda)G[p,q] > 0$.
Cancelling $\lambda(1-\lambda)$ yields

$$E[p] + E[q] - 2G[p,q] > 0. \tag{18}$$

The LHS of (18) equals $E[p-q]$ by calculation.
Note that $p \neq q \in L^2(S^2)$ implies $p - q \neq 0 \in L^2(S^2)$[8]. Since $E[q] > 0$ for $q \neq 0 \in L^2(S^2)$ when $\frac{\mu}{\beta} > 0$, $E[p-q] > 0$, which implies the inequality (16). □

3.4. **Existence of Stationary Points.** Using the standard variational method[4] to get the stationary points of E, which means we must find the Euler-Lagrange equation of E. Let $\Phi(\varepsilon) \equiv E[q+\varepsilon\eta]$. Setting

$$\begin{aligned} 0 &= \delta E \equiv \varepsilon \frac{d\Phi}{d\varepsilon}|_{\varepsilon=0} \\ &= \int_{S^2} \varepsilon\eta(\vec{x})d\vec{x}\Big[\int_{S^2} q(\vec{x'}) \ln \frac{1}{|1-\vec{x}\cdot\vec{x'}|} d\vec{x'} + 2\frac{\mu}{\beta}q(\vec{x})\Big] \end{aligned} \tag{19}$$

yields the Euler-Lagrange equation of E:

$$\int_{S^2} q(\vec{x'}) \ln \frac{1}{|1-\vec{x}\cdot\vec{x'}|} d\vec{x'} = (Gq)(\vec{x}) = -2\frac{\mu}{\beta}q(\vec{x}). \tag{20}$$

Expression (20) is the necessary condition for the existence of the stationary points under all values of $\frac{\mu}{\beta}$ with zero circulation. From *Remark 2*, we conclude that $-2\frac{\mu}{\beta} < 0$ is not the solution for equation (20). Therefore, the only solution for the Euler equation with zero circulation when $\frac{\mu}{\beta} > 0$ is the trivial one: $q \equiv 0$.

3.5. **The Unique Minimizer of $E[q]$.** Therefore, according to *Theorem 3.1*, $q \equiv 0$ is the unique minimizer of energy $E[q]$ with zero circulation when $\frac{\mu}{\beta} > 0$.

4. **The Extremal of $E[q]$ with Zero Circulation when $\frac{\mu}{\beta} < -\frac{||G||}{2}$.** In this section, we will get the extremal of $E[q]$ with zero circulation when $\frac{\mu}{\beta} < -\frac{||G||}{2}$. Let V be a reflexive Banach Space and $V^{''}$ a nonempty closed and bounded concave subset of V. Let F_2 be a functional which is bounded above and defined on $V^{''}$. It can be shown that:

Theorem 4.1. *(1) If F_2 is strictly concave, a necessary and sufficient condition that it has a unique maximizer is that its Gâteaux derivative equals to 0.*
(2) If F_2 is concave, a necessary and sufficient condition that it has a maximizer is that its Gâteaux equals to 0

Proof. See cf. Eckeland and Temam [3], Prop 1.2 on page 35 □

We know that the set of maximizers of E in $L^2(S^2)$, $maxE$, is a closed concave set which is non-empty by assumption. According to *Theorem 4.1, part(1)*, we will prove the uniqueness and existence of the maximizer of E in the following way:

(1) We will prove that $E[q] < 0$ for $q \neq 0 \in L^2(S^2)$ when $\frac{\mu}{\beta} < -\frac{||G||}{2}$.
(2) We will show $E[q]$ is strictly concave. Strict concavity of E in $maxE$ implies uniqueness of the maximizer of E.
(3)Since E is strictly concave in $L^2(S^2)$, stationary points of E must be unique maximizers of E and satisfy the equation (20).
(4) We observe that the only solution for (20) with zero circulation in $L^2(S^2)$ when $\frac{\mu}{\beta} < -\frac{||G||}{2}$ is $q \equiv 0$. Then we conclude that $q \equiv 0$ is the unique maximizer of E when $\frac{\mu}{\beta} < -\frac{||G||}{2}$ and $\int_{S^2} q(\vec{x})d\vec{x} = 0$.

4.1. E[q] is Bounded Above.

Lemma 4.2. *The function* $(Gq)(\vec{x}) = \int_{S^2} \ln \frac{1}{|1-\vec{x}\cdot\vec{x'}|} q(\vec{x'})d\vec{x'}$ *is in* $L^2(S^2)$

Proof. From the Cauchy-Schwarz inequality, we know
$||Gq(\vec{x})||_2^2 \leq \int_{S^2} q^2(\vec{x'})d\vec{x'} \int_{S^2} d\vec{x} \int_{S^2} d\vec{x'} \Big(\ln \frac{1}{|1-\vec{x}\cdot\vec{x'}|}\Big)^2 < \infty,$
which implies $(Gq)(\vec{x})$ is in $L^2(S^2)$.

□

Theorem 4.3. *E is negative definite in $L^2(S^2)$ when $\frac{\mu}{\beta} < -\frac{||G||}{2}$.*

Proof. (i) By *Lemma 4.2*, we know $H[q] = \frac{1}{2}\langle q, Gq\rangle \leq \frac{1}{2}||G||||q||^2$. When $\frac{\mu}{\beta} < -\frac{||G||}{2}$,

$$\begin{aligned} E[q] &\leq \frac{1}{2}||G||||q||^2 + \frac{\mu}{\beta}||q||^2 \\ &< \frac{1}{2}||G||||q||^2 - \frac{1}{2}||G||||q||^2 \\ &= 0. \end{aligned} \tag{21}$$

Therefore, $E[q] < 0$ for all $q \neq 0 \in L^2(S^2)$. (ii) From above proof part (1) and the fact $E[q \equiv 0] = 0$, we conclude: $E[q] \leq 0$ for all $q \in L^2(S^2)$ when $\frac{\mu}{\beta} < -\frac{||G||}{2}$. □

4.2. $E[q]$ is Strictly Concave.

Theorem 4.4. *$E[q]$ is strictly concave when $\frac{\mu}{\beta} < -\frac{||G||}{2}$.*

Proof. According to the definition of strict concavity, we only need to check that

$$E[\lambda p + (1-\lambda)q] > \lambda E[p] + (1-\lambda)E[q] \tag{22}$$

for all $p \neq q \in L^2(S^2)$, and $\lambda \in [0,1]$.
From the deduction in Section 3.3, we only need to check that $E[p-q] < 0$.
Since $E[q] < 0$ for $q \neq 0 \in L^2(S^2)$ when $\frac{\mu}{\beta} < -\frac{||G||}{2}$, we conclude $E[p-q] < 0$ for $p \neq q \in L^2(S^2)$. Therefore, $E[q]$ is strictly concave for all $q \in L^2(S^2)$ when $\frac{\mu}{\beta} < -\frac{||G||}{2}$. □

4.3. Existence of the Stationary Points.

Theorem 4.5. *$-2\frac{\mu}{\beta} \leq ||G||$ is a necessary condition for the existence of nonzero stationary points of the Euler equation (20) when $\frac{\mu}{\beta} < -\frac{||G||}{2} < 0$.*

Proof. All the stationary points of E must satisfy: $(Gq)(\vec{x}) = -2\frac{\mu}{\beta}q(\vec{x})$. When $\frac{\mu}{\beta} < -\frac{||G||}{2} < 0$, $||Gq|| = ||-2\frac{\mu}{\beta}q||$ implies $-2\frac{\mu}{\beta}||q|| \leq ||G||||q||$, which implies $-2\frac{\mu}{\beta} \leq ||G||$ for $q \neq 0 \in L^2(S^2)$.

□

By *Theorem 4.5*, we know that $-2\frac{\mu}{\beta} > ||G||$ here is not an eigenvalue of G.
Therefore, under the condition: $\frac{\mu}{\beta} < -\frac{||G||}{2}$, the only stationary point is the trivial one $q \equiv 0$.

4.4. Unique Maximizer of $E[q]$.

Therefore, according to *Theorem 4.1, part(1)*, $q \equiv 0$ is the unique maximizer of energy $E[q]$ with zero circulation when $\frac{\mu}{\beta} < -\frac{||G||}{2}$.

5. **Extremal of $E[q]$ with Zero Circulation when $\frac{\mu}{\beta} = -\frac{||G||}{2}$.** In this section, we want to get an extremal of $E[q]$ with zero circulation under the condition: $\frac{\mu}{\beta} = -\frac{||G||}{2}$.
According to *Theorem 4.1, part (2)*, we will prove an existence of the maximizer of E in the following way:

(1) We will prove that $E[q] \leq 0$ when $\frac{\mu}{\beta} = -\frac{||G||}{2}$.
(2) We will show $E[q]$ is concave.
(3) Since E is concave in $L^2(S^2)$, stationary points of E must be maximizers of E in $L^2(S^2)$ and satisfy the Euler equation (20).
(4) We observe that the only solution for (20) in $L^2(S^2)$ with zero circulation when $\frac{\mu}{\beta} = -\frac{||G||}{2}$ is the trivial one $q \equiv 0$. Then we conclude that $q \equiv 0$ is the maximizer of E with zero circulation when $\frac{\mu}{\beta} = -\frac{||G||}{2}$.

5.1. **$E[q]$ is Bounded Above.**

Theorem 5.1. *If $\frac{\mu}{\beta} = -\frac{||G||}{2}$, E is nonegative on $L^2(S^2)$*

Proof. (i) Since $H[q] \leq \frac{1}{2}||G||||q||^2$, therefore, when $\frac{\mu}{\beta} = -\frac{||G||}{2}$,
$E[q] \leq \frac{1}{2}||G||||q||^2 - \frac{||G||}{2}||q||^2 = 0$. (ii) From the fact $E[q \equiv 0] = 0$, we deduce that $E[q] \leq 0$ for $q \in L^2(S^2)$. □

5.2. **$E[q]$ is Concave.** First, we note that the expression $E[\lambda p+(1-\lambda)q]-\Big(\lambda E[p]-(1-\lambda)E[q]\Big) > 0(or \geq 0)$ is equivalent to the expression $E[p-q] < 0(or \leq 0)$, where $\lambda \in [0,1]$. From *Theorem 5.1*, we can only deduce that $E[p-q] \leq 0$ for $p, q \in L^2(S^2)$, which implies $E[q]$ is concave when $\frac{\mu}{\beta} = -\frac{||G||}{2}$.

5.3. **Existence of the stationary points.** From *Theorem 4.5*, we know that $-2\frac{\mu}{\beta} = ||G||$ could be the eigenvalue of G. From *Remark 3*, $||G||$ is the largest eigenvalue whose corresponding eigenfunction is $Y_{00} \in L^2(S^2)$[5]. But $\int_{S^2} Y_{00} = \sqrt{4\pi} > 0$ implies Y_{00} is not the stationary point of $E[q]$ with zero total circulation. Therefore, the only stationary point for zero circulation is the trivial one: $q \equiv 0$.

5.4. **Maximizer of $E[q]$.** According to *Theorem 4.1 part (2)*, $q \equiv 0$ is the maximizer of energy $E[q]$ with zero circulation when $\frac{\mu}{\beta} = -\frac{||G||}{2}$.

6. **Extremal of $E[q]$ with Zero Circulation when $-\frac{||G||}{2} < \frac{\mu}{\beta} < 0$.** In section 6 and 7, we will show the following results (1) with the exception of Y_{00}, all the spherical harmonics are stationary points of $E[q]$ with zero circulation when $-\frac{||G||}{2} < \frac{\mu}{\beta} < 0$; (2) there exist global maximizers of $E[q]$ with zero circulation; (3) $E[q]$ will neither bounded above nor bounded below in the certain specified intervals.
Existence of Stationary Points. When $-\frac{||G||}{2} < \frac{\mu}{\beta} < 0$, the stationary points of $E[q]$ also must satisfy the Euler equation (20). Since $||G||$ corresponds to the eigenfunction Y_{00} of G, the eigenspace of $||G||$ is one-dimensional [5], and since $0 < -2\frac{\mu}{\beta} < ||G||$, we conclude that all the spherical harmonics $\{Y_{lm}\}$ except Y_{00} are the stationary points.

7. **Unboundedness.** In this section, we want to derive the results that are related to Kraichnan's theory. Let λ_{lm} $(l = 1, 2 \ldots)$ and $\lambda_{l'm'}$ $(l' = 1, 2 \ldots)$ be the eigenvalues corresponding to Y_{lm} and $Y_{l'm'}$ of G. When $-\frac{||G||}{2} < \frac{\mu}{\beta} < 0$, we deduce the following results:

Lemma 7.1. *(1) If* $-\frac{\lambda_{lm}}{2} < \frac{\mu}{\beta} < -\frac{\lambda_{l'm'}}{2}$, $E[Y_{lm}] > 0 > E[Y_{l'm'}]$;
(2) If $-\frac{\lambda_{lm}}{2} < -\frac{\lambda_{l'm'}}{2}$, $E[Y_{lm}] > E[Y_{l'm'}]$.

Proof. (1) Since $E[q] = H[q] + \frac{\mu}{\beta}\Gamma[q] = \frac{1}{2}\langle q\ , Gq\rangle + \frac{\mu}{\beta}||q||^2$, we deduce

$$\begin{aligned} E[Y_{lm}] &= \frac{1}{2}\Big\langle Y_{lm}\ , G(Y_{lm})\Big\rangle + \frac{\mu}{\beta}||Y_{lm}||^2 \\ &= \frac{1}{2}\lambda_{lm} + \frac{\mu}{\beta} > 0. \end{aligned} \tag{23}$$

The last inequality follows from $-\frac{\lambda_{lm}}{2} < \frac{\mu}{\beta}$. We also deduce that $E[Y_{l'm'}] < 0$. Thus, $E[Y_{lm}] > 0 > E[Y_{l'm'}]$.
(2) From the proof of part (1) and $-\frac{\lambda_{lm}}{2} < -\frac{\lambda_{l'm'}}{2}$, we know that $E[Y_{lm}] - E[Y_{l'm'}] > 0$. □

Theorem 7.2. *Under the condition:* $-\frac{||G||}{2} < \frac{\mu}{\beta} < 0$,
(1) If $\frac{\mu}{\beta} = -\frac{\lambda_{lm}}{2}$ $(l = 1, 2 \ldots)$, $E[Y_{lm}] = 0$;
(2) Y_{10}, Y_{11}, $Y_{1,-1}$ *are the global maximizers of* $E[q]$ *with zero circulation*

Proof. (1)Since $\frac{\mu}{\beta} = -\frac{\lambda_{lm}}{2}$, we know that $E[Y_{lm}] = \frac{1}{2}\lambda_{lm} + \frac{\mu}{\beta} = 0$.
(2)From *Lemma 7.1 (2)* and the properties of $\{Y_{lm}\}$, we deduce that the stationary points Y_{10}, Y_{11}, $Y_{1,-1}$ satisfy:

$$E[Y_{10}] = E[Y_{11}] = E[Y_{1,-1}] > E[Y_{l'm'}], \tag{24}$$

where $l' = 2, 3 \ldots$, $m = 2l' + 1$. Therefore, Y_{10}, Y_{11}, $Y_{1,-1}$ are indeed the only global maximizers of $E[q]$ with zero circulation when $-\frac{||G||}{2} < \frac{\mu}{\beta} < 0$. □

Define $\lambda_{lm} > \lambda_{l+1,m'}$ $(l = 1, 2 \ldots)$ to be the adjacent eigenvalues of G.

Theorem 7.3. *When* $-\frac{||G||}{2} < \frac{\mu}{\beta} < 0$, *and* $\frac{\mu}{\beta}$ *satisfies:*
$-\frac{\lambda_{lm}}{2} < \frac{\mu}{\beta} < -\frac{\lambda_{l+1,m'}}{2}$ $(l = 1, 2, \ldots)$, *E[q] is neither bounded above nor bounded below.*

Proof. Let $\lambda_{l^*m^*}$ be the eigenvalue of G which corresponds to the eigenfunction $Y_{l^*m^*}$.
We prove this theorem in the following two parts:
(i) Assume that: $-\frac{\lambda_{l^*m^*}}{2} < \frac{\mu}{\beta}$. Therefore, $E[CY_{l^*m^*}] = C^2\big(\frac{1}{2}\lambda_{l^*m^*} + \frac{\mu}{\beta}\big)$, where C is a constant. We deduce that when C approaches infinity,

$$E[CY_{l^*m^*}] \longrightarrow +\infty. \tag{25}$$

(ii) Assume that $\lambda_{l^*m^*}$ satisfies: $-\frac{\lambda_{l^*m^*}}{2} > \frac{\mu}{\beta}$. When the constant C approaches infinity,

$$E[CY_{l^*m^*}] \longrightarrow -\infty. \tag{26}$$

From above part (i) and (ii), we conclude that $E[q]$ is neither bounded above nor bounded below when $-\frac{\lambda_{lm}}{2} < \frac{\mu}{\beta} < -\frac{\lambda_{l+1,m'}}{2}$. □

8. **Discussion.** The variational analysis of the energy-enstrophy functional E in this paper yields necessary conditions for the extremals of the E in the form of Euler-Lagrange equation (20). This Euler-Lagrange equation describes an integral eigenvalue problem in a compact way. We obtain a complete characterization of the global minimizer of E when $\beta > 0$ and global maximizer of E when $\beta < 0$. *Theorem 7.3* implies that the energy-enstrophy functional E is unbounded above and below when $-\frac{||G||}{2} < -\frac{\lambda_{lm}}{2} < \frac{\mu}{\beta} < -\frac{\lambda_{l+1,m'}}{2} < 0$. Because of this property, Kraichnan's theory[11], which is based on the Gibbs canonical probability: $P(q) = Z^{-1}e^{-\beta E[q]}$ is not well-defined in some physically significant regimes. In part due to this and other reasons, the first author introduced a spherical model[10] for energy-enstrophy theories where the enstrophy is constrained in a microcanonical way. The microcanonical constraint on enstrophy effectively removes the unboundedness property of the Gibbs canonical functional E.

REFERENCES

[1] Chjan Lim, James Montaldi, Mark Roberts, *Relative equilibria of point vortices on the sphere* Physica D 148 pages 97 - 135, 2001

[2] D. H. Griffel, *Applied functional analysis* Chichester, W. Sussex : E. Horwood ; New York : Halsted Press, 1981

[3] Ivar Ekeland, Roger Temam, *Convex analysis and variational problems* Amsterdam : North-Holland Pub. Co. ; New York : American Elsevier, 1976

[4] Richard Courant, D. Hilbert, *Volume 1,Methods of mathematical physics* John Wiley and Sons; December, 1953

[5] Frederick W. Byron, Jr. (and) Robert W. Fuller, *Mathematics of classical and quantum physics* Reading, Mass., Addison-Wesley Pub. Co., 1969

[6] A. J. Chorin, *Vorticity and turbulence* Springer-Verlag, New York, 1994

[7] L. Onsager, *Statistical hydrodynamics* Nuovo Cimento Spppl., 1949

[8] Gerald B. Folland, *Real analysis: modern techniques and their applications* New York : Wiley, 1984

[9] Paul K. Newton, *The N-vortex problem-analytical techniques* Springer-Verlag, New York, 2001

[10] Chjan C. Lim, *Coherent structures in an energy-enstrophy theory for axisymmetric Flows* Physics of Fluids 15, 478 - 487, 2003

[11] R. H. Kraichnan, *Statistical dynamics of two-dimensional flows* J Fluid Mech. 67, 155-175. 1975

[12] Colin J. Thompson, *Classical equilibrium statistical mechanics* Clarendon Pr, December, 1993

[13] Kimura, Y. (and) Okamoto, H., *Vortex motion on a sphere* J. Phys. Soc. Japan 56, 4203C4206. 1987

[14] T. D. Lee, *On some statistical properties of hydrodynamical and magnetohydrodynamical fields* Q. Appl. Math. 10, 69, 1952