Static stability and potential temperature

In the last problem set you derived the adiabatic lapse rate of a dry atmosphere:

$$\frac{dT}{dz} = -\frac{g}{c_p} \equiv \Gamma_d$$  \hspace{1cm} (1)

In class we then derived an expression for the temperature that an air parcel at altitude \(z\) (with temperature \(T_z\) and pressure \(p_z\)) would have if brought down to the surface adiabatically. This “potential temperature” \(\theta\) at any altitude \(z\) is given by

$$\theta_z = T_z \cdot \left(\frac{p_s}{p_z}\right)^{(\gamma-1)/\gamma}$$  \hspace{1cm} (2)

where \(p_s\) is the surface pressure.

If the atmospheric temperature profile follows the dry adiabatic lapse rate, then the atmosphere is “neutrally stable”, and air from any level can readily move to any other level. In this case the potential temperature is constant throughout the atmosphere, and simply equals the surface temperature: \(\theta_z = T_s\).

If the atmosphere has a positive gradient in potential temperature, the atmospheric temperature profile is “statically stable”. If air from altitude were brought to the ground (and surface pressure), then that air would be warmer than its surroundings, and therefore less dense, and therefore buoyant, and so would rise again. Air with higher potential temperature therefore “wants” to sit on top of air with lower potential temperature. The gradient in potential temperature tells you whether an atmospheric temperature profile is unstable, neutrally stable, or statically stable.
Enthalpy

In class we defined the “enthalpy” of a system as
\[ H = U + pV \]
which for an ideal gas is
\[ H = C_v T + pV \]
where \( C_v \) has units of J/K. The enthalpy is the internal energy of a system plus a term that relates to the work done as pressure is changed. Converting to per mass units (dividing by \( N \cdot \overline{M} \)) gives
\[ h = c_v T + \frac{p}{\rho} \]
since density \( \rho \) is mass/volume or \( N \overline{M}/V \). Using the ideal gas law \( (p = \rho \cdot R^* T \) or \( p/\rho = R^* T \)) this simplifies to
\[ h = (c_v + R^*) \cdot T = c_p T \]
entirely consistent with the definition of \( C_p = \left( \frac{\partial H}{\partial T} \right)_p \).

1 Intuitive thinking about stability

Imagine that an atmosphere initially has a dry adiabatic temperature profile (and imagine that this world has no moisture). The atmosphere is therefore initially neutrally stable.

(A) Draw a plot of \( T \) vs. altitude \( z \) and draw the dry adiabatic temperature profile, with an initial surface temperature \( T_{s0} \).

(B) Overnight, the surface radiatively cools to space, so that air immediately above the surface becomes colder than \( T_{s0} \). At higher altitudes, the air is optically thin in the infrared and so cannot cool much overnight, so its temperature doesn’t change much. If the surface cooling is severe, it can even produce a temperature inversion: temperature near the ground is colder than temperature higher up. Draw this new temperature profile on your plot. Does the cooling make the atmosphere more or less stable?

(C) During the day, the surface is warmed by absorbing sunlight, and so the air immediately above the surface becomes warmer. Eventually that warming is so large that the surface temperature becomes warmer than \( T_{s0} \). Draw this as well. Does that warming make the atmosphere more or less stable?

(D) Based on the above, at what time of day would you imagine convective activity (storm formation) is strongest? Answer first, then compare your answer to the timing of hail reported here [link](http://www.srh.noaa.gov/images/meg/research/svrclim/svrfig7.gif) by the National Weather Service office in Memphis.
(E) What would you infer about vertical air motions in the stratosphere?
2 Relating enthalpy to atmospheric stability

Many people asked, quite reasonably, what the concept of enthalpy is useful for. This question provides one use of enthalpy. We can use enthalpy to define the “dry static energy” of an air parcel:

\[ s = h + gz \]  

(4)

The dry static energy has units of Joules and is the sum of an air parcel’s internal energy, the work term, and its gravitational potential energy \( gz \). You might imagine that just like potential temperature, dry static energy is a conserved quantity for air being raised or lowered adiabatically. Prove that. That is, show that in an atmosphere with a dry adiabatic lapse rate:

(A) Dry static energy is constant throughout the atmosphere (\( ds/dz = 0 \)).

(B) The dry static energy is proportional to the potential temperature: \( s = c_p \theta \).

(Hint: you know that in an atmosphere whose temperature profile is dry adiabatic, \( \theta = T_s \) everywhere. That’s the definition of the dry adiabat.)

3 Graduate students or for extra credit, do the following

Enthalpy and potential temperature in non-adiabatic atmospheres

If an atmosphere has a dry adiabatic lapse rate, you found above that it is exactly true that \( s = c_p \theta \). In a non-adiabatic atmosphere that doesn’t hold. But, it’s still a pretty good approximation. We can at least say that \( s \sim c_p \theta \) in the lower part of the troposphere.

Show that this holds even in an extremely non-adiabatic case. Consider a fully isothermal atmosphere, so that \( T_z = T_s \) for all \( z \).

(A) First, to simplify the definition of \( \theta \), show that the exponent equals \( R^* / c_p \).

(B) Show that for this atmosphere, \( s \sim c_p \theta \) when \( z \) is small.

(C) Estimate the altitude \( z \) below which \( s \) and \( c_p \theta \) differ by no more than 1%.