1 Climate phenomenology (continued)

Look at the movie of MODIS reflectivity posted at [http://vimeo.com/106645463](http://vimeo.com/106645463) that shows the motions of clouds across the Earth. This animation lets you see faster air motions than are possible in an animation of monthly means.

**Wind directions.** What direction does air move in the following regions?

(A) Northern midlatitudes  
(B) Southern midlatitudes  
(C) Equatorial region

Terminology: if winds are from the west, they are “westerlies”, if from the east, “easterlies”. If you are saying that something moves toward the west, it is going “westward”; toward the east, “eastward”.

**Hurricanes.** Watch til you see a hurricane or tropical storm in the following areas. When you do, hit pause and grab a screenshot, print it, circle the hurricane, and include it in your problem set.

(D) Atlantic ocean. These spin off of W. Africa  
(E) Pacific ocean. These develop in the W. Pacific (right side of screen)  
(F) Describe the typical trajectories that storms travel in each case. How does that relate to your answers in Problem 1?  
(G) *(Optional)* If you want to cross-country ski in Chicago in the winter you’re usually out of luck, but if you drive to the other side of Lake Michigan, in Michigan, you often find snow. Why?
2 Mean planetary temperatures with realistic geometry

In class we calculated the effective emission temperature of the Earth \( T_E \): the temperature required for blackbody emission to exactly balance mean incoming solar radiation \( \bar{F} \), that is:

\[
\bar{F} = \sigma T_E^4
\]

The calculation of \( T_E \) effectively assumes the entire planet has the same temperature, i.e. it represents this simple model, of an Earth uniformly illuminated over its entire surface:

![Figure 2.1: Model effectively used to estimate \( T_E = 255 \) K.](image)

We used an incoming flux \( \bar{F} = 235 \) of W/m\(^2\) (derived by taking the Earth’s albedo as \( A \sim 0.3 \), or 30% of incoming solar radiation reflected away), and got an emission temperature of \( T_E = 255 \) K.

In some textbooks, this hypothetical \( T_E \) (255 K) is compared to the actual mean surface temperature \( T_s \) (287 K), and the authors then say, the difference (287 - 255 = 32 degrees) shows you the warming effect of the atmosphere. But that’s an inappropriate comparison. It’s comparing a hypothetical emission temperature of a uniformly illuminated model Earth with the spatially average temperature of a real Earth that is not at all uniformly illuminated. The real Earth gets more insolation at the equator than the poles, and is hotter at the equator than the poles. (This temperature gradient is somewhat moderated by the Earth’s considerable equator-pole heat transport, which means that individual locations are not in radiative balance with local incoming solar radiation.)

To make a proper assessment of “what the atmosphere does” for climate, you’d have to compare apples to apples or oranges to oranges. You’d have to compare average surface temperature for a uniformly illuminated Earth without (255 K) and with (??) an atmosphere, or to compare surface temperature for a realistically illuminated Earth without (??) and with (287 K) an atmosphere.

In class I argued that if you did the geometry properly, the mean temperature in the no-atmosphere case would be lower than \( T_E = 255 \) K. If every point on the Earth is in radiative equilibrium with its incoming flux, then

\[
\overline{\sigma \cdot T^4} = \bar{F} = \sigma \cdot T_E^4
\]

where \( T \) is the temperature at each location on Earth and the overbar indicates an average. But the average of (temperature to the fourth power) is not the same as the (average temperature) raised to the fourth power: \( \overline{T^4} \neq \overline{T}_E^4 \). So the effective emission temperature \( T_E \) that we defined is not the same as the mean Earth temperature \( \bar{T} \).

In the problems below, convince yourself that the true average temperature \( \bar{T} \) of a no-atmosphere Earth is always colder than \( T_E \) if temperature varies across the planet.
First, do a simple model of a two-dimensional Earth. Let the Earth be a flat plate, and calculate its mean temperature in two cases:

(A) The plate spins so rapidly that both sides can be considered equally illuminated (Figure 2.2). (That is, the night side doesn’t have time to cool down.)

(B) The plate stays fixed relative to the sun, with one side illuminated and the other remaining in darkness (Figure 2.3). Besides the mean temperature, give also the temperature of the illuminated side.

(C) Is the mean temperature lower when planetary temperature is uniform or when it varies spatially?

Then, do a more realistic calculation that admits that the Earth is a sphere. Assume the normal Earth albedo ($A \sim 0.3$) so that you can compare each calculation to your previously derived $T_E$.

As a first step, assume a rapidly spinning three-dimensional Earth (Figure 2.4). The poles are less illuminated than the equator, but you can assume that the night is so short that there’s no day/night difference in temperature, so that at a given latitude $\theta$ every point around the Earth has the same temperature.

Finally, you’ll do a more realistic model of a slowly-spinning Earth with a dark nightside (Figure 2.5). Now you are assuming that every point on the Earth’s surface is in radiative equilibrium with incoming solar radiation, i.e. that the outermost surface of the planet would adjust quickly as its insolation changed.

(D) What is the maximum temperature of the fast-spinning planet of Fig. 2.4? Put your number in context. (If you need to convert to Celsius or Fahrenheit for an intuitive comparison to the real world, do that.)
(E) What is the mean temperature of this planet? Greater or less than $T_E$?

(F) What is the maximum temperature of the slow-spinning planet of Fig. 2.5?

(G) What is the mean temperature of this planet? Greater or less than $T_E$? Greater or less than the more uniform planet in Figure 2.4?

For these problems you’ll have to integrate temperature over a sphere or hemisphere and then divide by area. In the process you will run into an integral you probably don’t want to tackle. It’s fine to just numerically integrate it. There are many good online tools for this purpose, including one from the makers of Mathematica: [http://www.wolframalpha.com/input/?i=integral](http://www.wolframalpha.com/input/?i=integral). (Click on “also include domain of integration” to put in the limits.)

The model that you made above (Figure 2.5) is basically the moon. The moon is a real-world analogue for what the Earth would look like with no atmosphere: it’s a rocky planetoid at the same distance from the sun as the Earth, but with zero atmosphere. And conveniently, the moon’s temperature has actually been measured (Figure 2.6). The only important difference to take into consideration is that the moon is less reflective than Earth, with an albedo of only $A \sim 0.07$ (7% of sunlight reflected away). The moon may look shiny in the sky, but without clouds, ice, or snow, it absorbs more of its incoming sunlight than does the Earth. So you’d rescale the incoming flux you used above by $(1 - A_{Moon})/(1 - A_{Earth})$.

**Diviner-Measured Moon Equatorial Temperatures**

Data from Ashwin Vasavada at NASA (March 27, 2012)

![Figure 2.6: Moon temperature at the equator.](image)

(H) Check: is the maximum moon temperature at the equator consistent with the maximum temperature you calculated above in (F)?

(I) Is the diurnal cycle of moon temperatures consistent with what you’d expect? Discuss, and be grateful the Earth has an atmosphere.
3 Planck’s function and blackbody radiation

“The Planck function is worthy of respect, if not awe, in that it contains not one, not two, but three fundamental (or at least believed to be so) constants of nature: the speed of light in a vacuum c, Planck’s constant \( h \), and Boltzmann’s constant \( k_B \). You can’t get much more fundamental than that.”


The Planck function gives the irradiance of a blackbody as a function of frequency (or, alternatively, wavelength), i.e. the total power emitted in a frequency interval, in units of \( \text{W/m}^2/(1/s) = (\text{W/m}^2) \cdot \text{s} \).

\[
B(\nu, T) \, d\nu = \frac{2\nu^3}{c^2} \frac{1}{(e^{\nu/k_B T} - 1)}
\]

The total irradiance across all frequencies \( F \), in units of \( \text{W/m}^2 \) is found by integrating the Planck function.

\[
F = \int_0^\infty B(\nu, T) \, d\nu = \sigma \cdot T^4
\]

where \( \sigma \), the Stefan-Boltzmann constant, just tidies up the constants: \( \sigma = \frac{2\pi^5 k_B^4}{15 \hbar^3 c^2} = 5.67 \times 10^{-8} \text{W m}^{-2} \text{K}^{-4} \). This is an ugly integration so I left it as an optional problem to actually do it analytically.

You can see many relevant properties of the Planck function just by inspection. Explain (formal math is OK, but you can also just talk about the behavior of individual terms) why each of the following are true:

(A) irradiance goes to zero at high frequency (\( \nu \) large)

(B) irradiance goes to zero at very low frequency (\( \nu \) small)

Note that if both the above are true, irradiance must have a maximum at some intermediate frequency.

(C) irradiance at every frequency is larger when \( T \) is larger

(D) the maximum of the function should shift to higher frequency when \( T \) is larger

Note that temperature appears in the Planck function only in the exponent \( \hbar \nu/k_B T \). The numerator \( \hbar \nu \) is the energy of a photon of frequency \( \nu \). The denominator \( k_B T \) is a measure of the thermal energy of matter. The ratio therefore relates in some way to the ability of matter to emit photons of a particular energy. The fact that photons are quantized means that matter can only emit light of frequency \( \nu \) if it can manage to produce a photon of energy \( \hbar \nu \). If that necessary energy for producing a photon were large relative to the material’s thermal energy \( (\hbar \nu << k_B T) \), you would not expect that many photons could be produced. On the other hand, if thermal energy were high relative to the energy of a photon at a given frequency \( (\hbar \nu >> k_B T) \), you’d expect that these photons could be readily emitted.

Find the peak of the Planck function (in frequency) for different temperatures. It can be useful to define the variable \( u = \hbar \nu/k_B T \) for this purpose. The maximum of the Planck function in these units occurs near \( u \sim 2.82 \).

(E) Write the Planck function as \( B(u, T) \), i.e. as a function of \( T \), physical constants, and \( u \).

\[\footnote{But see Bohren p. 10 for a caution: “Failure to recognize that the maximum of a distribution function depends on how it is plotted has led and no doubt will continue to lead to errors.” The peak is different if \( B \) is expressed in terms of wavelength.}\]
State what frequency the peak irradiance corresponds to in the following cases, and discuss. (If the peak frequency is in the visible, say what color it is).

(F) the mean temperature of the Earth’s surface (287 K)
(G) iron in a blacksmith’s fire (1000 K)
(H) a lightbulb filament (3000 K)
(I) a halogen bulb (3400 K)
(J) the temperature of the sun’s photosphere (5800 K)

Numerically integrate the Planck function (see link to online integration tool in Problem 2) to determine

(K) For a red-hot horseshoe at the blacksmith’s (1000 K), what fraction of its emitted energy is visible light? (λ between 400−700 nm)

(L) For an incandescent lightbulb (3000 K), what fraction of its emitted energy is at wavelengths too long for the eye to see? (> 700 nm). Is this consistent with the (in)efficiency of incandescent bulbs at converting electrical power to visible light? (typically <10%)

(M) For a halogen bulb (3400 K), what fraction of its emitted energy is dangerous ultraviolet? (λ < 400 nm)

Optional problems. Grad students should do these, and physics majors should at least do N and O.

The “classical limit” of the Planck function is when the quantized energy of the photon does not affect radiation: that is, when \( u = h\nu/k_B T \) is very small.

(N) Use that assumption to simplify the Planck function. Show that in your simplified function, irradiance becomes linear with temperature \( B(\nu, T) \propto T \).

(O) Now assume you then forgot somehow that quantum physics existed and mistakenly believed that your function above was valid in all cases. Would that be physically possible? Explain why you produced an “ultraviolet catastrophe”.

(P) Differentiate \( B(u, T) \) to prove that the peak occurs at \( u \sim 2.82 \). (Or even just plot \( B(u, T) \) vs \( u \) and estimate by hand).

(Q) Integrate the Planck function to derive the Stefan-Boltzmann law. (This is tricky and you may need to consult textbooks).