Appendix A: Derivation of the Planck Function

In accordance with Boltzmann statistics, if \( N_0 \) denotes the number of oscillators in any given energy state, then the number \( N \) in a state having energy higher by an amount \( \varepsilon \) is given by

\[
N = N_0 e^{-\varepsilon/KT},
\]

(A.1)

where \( K \) is Boltzmann’s constant and \( T \) is the absolute temperature. On the basis of Planck’s first postulation, an oscillator cannot have just any energy, but rather only energies given by Eq. (1.2.1). Thus, the possible values of \( E \) must be \( 0, h\tilde{\nu}, 2h\tilde{\nu}, \) and so on. If the number of oscillators with zero energy is \( N_0 \), then by virtue of Eq. (A.1), the number with energy \( h\tilde{\nu} \) is \( N_0 e^{-h\tilde{\nu}/KT} \), the number with energy \( 2h\tilde{\nu} \) is \( N_0 e^{-2h\tilde{\nu}/KT} \), and so on. The total number of oscillators with frequency \( \tilde{\nu} \) for all states is therefore

\[
N = N_0 + N_0 e^{-h\tilde{\nu}/KT} + N_0 e^{-2h\tilde{\nu}/KT} + \ldots
\]

\[
= N_0 \left[ 1 + e^{-h\tilde{\nu}/KT} + (e^{-h\tilde{\nu}/KT})^2 + \ldots \right]
\]

\[
\cong N_0 / (1 - e^{-h\tilde{\nu}/KT}).
\]

(A.2)

The total energy of these oscillators may be obtained by multiplying each term in Eq. (A.2) by the appropriate energy level:

\[
E = 0 \cdot N_0 + h\tilde{\nu} \cdot N_0 e^{-h\tilde{\nu}/KT} + 2h\tilde{\nu} \cdot N_0 e^{-2h\tilde{\nu}/KT} + 3h\tilde{\nu} \cdot N_0 e^{-3h\tilde{\nu}/KT} + \ldots
\]

\[
= h\tilde{\nu} N_0 e^{-h\tilde{\nu}/KT} \left[ 1 + 2e^{-h\tilde{\nu}/KT} + 3(e^{-h\tilde{\nu}/KT})^2 + \ldots \right]
\]

\[
\cong h\tilde{\nu} N_0 e^{-h\tilde{\nu}/KT} / (1 - e^{-h\tilde{\nu}/KT})^2.
\]

(A.3)

The average energy per oscillator is then given by

\[
\frac{E}{N} = \frac{N_0 h\tilde{\nu} e^{-h\tilde{\nu}/KT} / (1 - e^{-h\tilde{\nu}/KT})^2}{N_0 / (1 - e^{-h\tilde{\nu}/KT})} = h\tilde{\nu} / (e^{h\tilde{\nu}/KT} - 1).
\]

(A.4)

According to Planck’s second postulation, the quanta of energy are emitted only when an oscillator changes from one to another of its quantized energy states. The average emitted energy of a group of oscillators is, therefore, given by Eq. (A.4), which is the factor appearing in Planck’s formula.
To obtain the Planck function, we let $u_{\tilde{\nu}}$ denote the monochromatic energy density, i.e., the energy per unit volume per unit frequency interval in a cavity with temperature $T$. With this definition, we write

$$u_{\tilde{\nu}} = A h_{\tilde{\nu}} / (e^{h_{\tilde{\nu}}/kT} - 1), \quad (A.5)$$

where $A$ is a constant to be determined. In accordance with the principle of equipartition of energy, the energy density in a cavity is given by the classical Rayleigh–Jeans formula:

$$u_{\tilde{\nu}} = (8\pi \tilde{\nu}^2 / c^3) KT. \quad (A.6)$$

This formula is valid when the temperature $T$ is high and the frequency $\tilde{\nu}$ is small. By letting $h_{\tilde{\nu}} / kT \rightarrow 0$ in Eq. (A.5), we find $A = 8\pi \tilde{\nu}^2 / c^3$. Thus, the monochromatic energy density is

$$u_{\tilde{\nu}} = \frac{8\pi h_{\tilde{\nu}}^3}{c^3(e^{h_{\tilde{\nu}}/kT} - 1)}. \quad (A.7)$$

For blackbody radiation, the emitted photons travel in all directions (4$\pi$ solid angle) at the speed of light $c$. Thus, the emitted intensity (or radiance) in a cavity with a temperature $T$ in units of energy/area/time/sr/frequency may be expressed by

$$B_{\tilde{\nu}}(T) = u_{\tilde{\nu}} c / 4\pi. \quad (A.8)$$

On substituting Eq. (A.7) into (A.8), we obtain the Planck function in the form

$$B_{\tilde{\nu}}(T) = \frac{2h_{\tilde{\nu}}^3}{c^2(e^{h_{\tilde{\nu}}/kT} - 1)}, \quad (A.9)$$

as stated in Eq. (1.2.3).